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Citation for published version (APA):

Wigman, I., & Yesha, N. (Accepted/In press). CLT for mass distribution of toral Laplace eigenfunctions. *MATHEMATIKA*.

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CLT FOR PLANCK SCALE MASS DISTRIBUTION OF TORAL LAPLACE EIGENFUNCTIONS

IGOR WIGMAN AND NADAV YESHA

ABSTRACT. We study the fine scale L^2 -mass distribution of toral Laplace eigenfunctions with respect to random position, in 2 and 3 dimensions. In 2d, under certain flatness assumptions on the Fourier coefficients and generic restrictions on energy levels, both the asymptotic shape of the variance is determined and the limiting Gaussian law is established, in the optimal Planck-scale regime. In 3d the asymptotic behaviour of the variance is analysed in a more restrictive scenario (“Bourgain’s eigenfunctions”). Other than the said precise results, lower and upper bounds are proved for the variance, under more general flatness assumptions on the Fourier coefficients.

1. INTRODUCTION

Given a smooth compact d -manifold \mathcal{M} we are interested in the spectral properties of the Laplace-Beltrami operator Δ on \mathcal{M} . It is well-known that the eigenvalue spectrum of Δ is purely discrete, i.e., the set of numbers E admitting a solution to the Helmholtz equation

$$\Delta\phi + E\phi = 0$$

is a sequence $\{E_j\}_{j \geq 1}$ of numbers ordered with multiplicity in a non-decreasing order such that $E_j \rightarrow \infty$. We denote the corresponding sequence $\{\phi_j\}_{j \geq 1}$ of (real-valued) eigenfunctions constituting an orthonormal basis of the square-integrable functions $L^2(\mathcal{M})$ on \mathcal{M} ; the sequence $\{\phi_j\}_{j \geq 1}$ is uniquely determined up to the spectral degeneracies (i.e., up to orthogonal transformations in each eigenspace of dimension ≥ 2).

1.1. Shnirelman’s Theorem and Small-Scale Equidistribution. Assuming w.l.o.g. that \mathcal{M} is unit volume $\text{Vol}(\mathcal{M}) = 1$, the celebrated Shnirelman’s Theorem [28, 30, 10] asserts that if \mathcal{M} is chaotic (i.e., the geodesic flow on \mathcal{M} is ergodic), then “most” of the $\{\phi_j\}$ are L^2 -equidistributed. In particular, they are equidistributed in position space, i.e., there exists a density 1 sequence j_k such that for all “nice” domains $\mathcal{A} \subseteq \mathcal{M}$ we have

$$(1.1) \quad \lim_{k \rightarrow \infty} \int_{\mathcal{A}} \phi_{j_k}(x)^2 dx = \text{Vol}(\mathcal{A}).$$

Beyond Shnirelman’s Theorem, Berry’s universality conjecture [4, 5] implies that for a *generic* chaotic manifold (1.1) holds for \mathcal{A} shrinking with k , slower than the Planck’s scale $E_{j_k}^{-1/2}$. More precisely, it states that there exists a density 1 sequence $\{j_k\}_k$ so that if $r_0(E) : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ satisfies $r_0(E) \cdot E^{1/2} \rightarrow \infty$ diverging arbitrarily slowly, then, for $B_x(r)$ the radius r geodesic ball in \mathcal{M} centred at x , we have

$$(1.2) \quad \left| \int_{B_x(r)} \phi_{j_k}(y)^2 dy - \text{Vol}(B_x(r)) \right| = o_{k \rightarrow \infty}(r^d)$$

uniformly for all $x \in \mathcal{M}$ and $r > r_0(E_{j_k})$, i.e.,

$$(1.3) \quad \sup_{\substack{x \in \mathcal{M} \\ r > r_0(E_{j_k})}} \left| \frac{\int_{B_x(r)} \phi_{j_k}(y)^2 dy}{\text{Vol}(B_x(r))} - 1 \right| \rightarrow 0.$$

The following recent results are rigorous manifestations of the small-scale (“shrinking balls”) statement (1.3). Luo and Sarnak [25, Theorem 1.2] established the small-scale equidistribution for Laplace eigenfunctions on the modular surface (assuming in addition that they are Hecke eigenfunctions) where $r > E^{-\alpha}$ with a small $\alpha > 0$, and Young [29], conditionally on GRH, refined this estimate for $r > E^{-1/6+o(1)}$ holding for *all* such eigenfunctions. Hezari and Rivière [19], and independently Han [16] established the equidistribution for Laplace eigenfunctions on manifolds of negative curvature on logarithmic scale (i.e., $r > (\log E)^{-\alpha}$, for some $\alpha > 0$), and Han [17] considered random Laplace eigenfunctions on “symmetric” manifolds, of high spectral degeneracy; here the higher the spectral degeneracy is the smaller the allowed scale is. More recently, Han and Tacy [18] proved small-scale equidistribution for random Gaussian combinations of eigenfunctions on compact manifolds for $r > E^{-1/2+o(1)}$, and de Courcy-Ireland [11] showed that, with high probability, the L^2 -mass of random Gaussian spherical harmonics is, up to a small error, equidistributed, slightly above Planck scale.

1.2. Toral Laplace eigenfunctions. For the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$, there are high spectral degeneracies; in this case Lester and Rudnick [24, Theorem 1.1] proved that the small-scale equidistribution is satisfied by a generic Laplace eigenfunction (also considered by Hezari and Rivière [20]). More precisely, they showed that every o.n.b. $\{\phi_j\}$ admits a density one subsequence $\{\phi_{j_k}\}$ of Laplace eigenfunctions obeying (1.3), with $r_0(E) = E^{-\alpha(d)}$, where $\alpha(d)$ is any number smaller than

$$(1.4) \quad \alpha(d) < \frac{1}{2(d-1)},$$

an (almost) optimal Planck-scale result for $d = 2$, yet somewhat weaker than Berry’s conjecture for $d > 2$.

One can express the real toral Laplace eigenfunctions explicitly as a sum of exponentials

$$(1.5) \quad f_n(x) = \sum_{\lambda \in \mathcal{E}_n} c_\lambda e(\langle x, \lambda \rangle), \quad (c_{-\lambda} = \overline{c_\lambda})$$

for

$$(1.6) \quad n \in S_d := \{n = a_1^2 + \dots + a_d^2 : a_1, \dots, a_d \in \mathbb{Z}\}$$

expressible as a sum of d integer squares, and the corresponding frequencies λ are the standard lattice points

$$(1.7) \quad \mathcal{E}_n = \mathcal{E}_{d;n} = \{\lambda \in \mathbb{Z}^d : \|\lambda\|^2 = n\}$$

lying on the $(d-1)$ -dimensional sphere (a circle for $d = 2$) of radius- \sqrt{n} ; in this case the energy is $E = E_n = 4\pi^2 n$. We will assume w.l.o.g. that f_n is L^2 -normalised, equivalent to

$$(1.8) \quad \|f_n\|_{L^2(\mathbb{T}^d)}^2 = \sum_{\lambda \in \mathcal{E}_n} |c_\lambda|^2 = 1.$$

For every $n \in S_d$, denote

$$(1.9) \quad N = N_{d;n} = \#\mathcal{E}_n.$$

When $d = 2$, by Landau's theorem, $\{n \leq x : n \in S_2\} \sim K \frac{x}{\sqrt{\log x}}$ where $K > 0$ is the “Landau-Ramanujan constant”. On average $N = N_{2;n}$ is of order of magnitude $\sqrt{\log n}$; however, for a density one sequence in S_2 we have $N = (\log n)^{\log 2/2 + o(1)}$. In general, for $n \in S_2$ we have

$$N = n^{o(1)}.$$

For $d = 3$, Siegel's theorem asserts that for $n \not\equiv 0, 4, 7 \pmod{8}$,

$$N = N_{3;n} = n^{1/2 + o(1)};$$

since $x \mapsto 2^a x$ is a bijection between the solutions to $x_1^2 + x_2^2 + x_3^2 = n$ and $x_1^2 + x_2^2 + x_3^2 = 4^a n$, we can always assume that $n \not\equiv 0, 4, 7 \pmod{8}$ with no loss of generality.

Granville and Wigman [15, Theorem 1.2] refined the aforementioned estimate by Lester-Rudnick for $d = 2$. They proved that in this case, (1.3) is valid slightly above Planck-scale $r_0(E) = E^{-1/2 + o(1)}$, for *all* eigenfunctions f_n as in (1.5), corresponding to numbers n so that the lattice points \mathcal{E}_n are well-separated (“Bourgain-Rudnick sequences”), a condition satisfied [9, Lemma 5] by “generic” integers $n \in S_2$ in a strong quantitative sense, subsequently refined in [15, Theorem 1.4], see section 2.2.

1.3. Averaging mass w.r.t. ball centre. For both the 2-dimensional and the higher-dimensional tori it is possible to construct exceptional examples of sequences of toral eigenfunctions where the equidistribution condition is not satisfied: for $d \geq 2$ thin sequences [24, Theorem 3.1] $\{\phi_{j_k}\}$ of eigenfunctions violating condition (1.2) at Planck-scale $r \cdot E_{j_k}^{1/2} \rightarrow \infty$, around the origin $x = 0$, and even stronger, for $d \geq 3$ [24, Theorem 4.1 (construction by J. Bourgain)] eigenfunctions violating (1.2) with $r \gg E^{-\alpha(d)}$ where $\alpha(d) > \frac{1}{2(d-1)}$, again around the origin $x = 0$. In these cases, rather than keeping the ball centre $x = 0$ at the origin, one may vary x , and study whether the “typical” discrepancy on the l.h.s. of (1.2) is *small*, even if the existence of x so that the l.h.s. of (1.2) is *not small* is known, so that, in particular, (1.3) is not satisfied.

A natural way to vary x is to think of x as *random*, drawn uniformly in \mathbb{T}^d . We define the random variable

$$(1.10) \quad X_{f_n, r} = X_{f_n, r; x} := \int_{B_x(r)} f_n(y)^2 dy,$$

and are interested in the distribution of $X_{f_n, r}$ where x is drawn randomly uniformly in \mathbb{T}^d . The relevant moments are: expectation

$$(1.11) \quad \mathbb{E}[X_{f_n, r}] = \int_{\mathbb{T}^d} X_{f_n, r; x} dx,$$

higher centred moments

$$(1.12) \quad \mathbb{E}[(X_{f_n, r} - \mathbb{E}[X_{f_n, r}])^k] = \int_{\mathbb{T}^d} (X_{f_n, r; x} - \mathbb{E}[X_{f_n, r}])^k dx, \quad k \geq 2,$$

and in particular the variance

$$(1.13) \quad \mathcal{V}(X_{f_n, r}) = \mathbb{E}[(X_{f_n, r} - \mathbb{E}[X_{f_n, r}])^2].$$

This approach of averaging the L^2 -mass with respect to the ball centre (and keeping f_n fixed) was pursued by Granville-Wigman [15] in the 2-dimensional case, again slightly above the Planck scale $r > E^{-1/2 + o(1)}$. In this regime, by proving an upper bound for $\mathcal{V}(X_{f_n, r})$ beyond $(\mathbb{E}[X_{f_n, r}])^2 = O(r^4)$, valid for *all* $n \in S_2$, under some flatness assumption on f_n (cf. Definition 1.4 below), they established (1.2) for “typical”, if *not all* $x \in \mathbb{T}^2$. It would be desirable to find a regime where it is possible to analyse the precise asymptotic behaviour of the variance $\mathcal{V}(X_{f_n, r})$ of $X_{f_n, r}$, and, if possible, determine the limit

distribution law for $X_{f_n, r}$; our principal results below achieve both of these in the 2-dimensional case, and the former in the 3-dimensional one (see theorems 1.1 and 1.3). Such an approach of bounding the discrepancy variance while averaging over ball centres was recently used by Sarnak [26] for mass distribution of forms on symmetric spaces, and P. Humphries [21] for mass distribution of automorphic forms.

1.4. Statement of the main results for $d = 2, 3$: asymptotics for the variance, CLT. Our principal results below are applicable to “flat” functions for $d = 2, 3$, understood in suitable, more and less restrictive, senses. For example, “Bourgain’s eigenfunction” [7]

$$(1.14) \quad f_n(x) = \frac{1}{\sqrt{N}} \sum_{\lambda \in \mathcal{E}_n} \varepsilon_\lambda e(\langle x, \lambda \rangle)$$

with $\varepsilon_\lambda = \pm 1$ for every $\lambda \in \mathcal{E}_n$, i.e. corresponding to $|c_\lambda| = N^{-1/2}$, satisfies any of the flatness conditions in the most restrictive sense. Denote \mathcal{B}_n to be the class of Bourgain’s eigenfunctions.

Our first principal result determines the precise asymptotic behaviour of the variance $\mathcal{V}(X_{f_n, r})$ for the 2-dimensional case, and moreover asserts that the moments of the standardized random L^2 -mass of f_n are asymptotically Gaussian; we subsequently deduce a Central Limit Theorem (see Corollary 1.2). For the sake of elegance of presentation, it is formulated for Bourgain’s eigenfunctions (1.14); below we formulate a more general result which holds for a larger class of flat eigenfunctions (see Theorem 2.5 in section 2), and later a result where the averaging over the ball centre x is itself restricted to shrinking balls (Theorem 8.3 in section 8).

Theorem 1.1 (Gaussian moments, $d = 2$, Bourgain’s eigenfunctions). *There exists a density one sequence $S'_2 \subseteq S_2$ so that the following holds. Let $r_0 = r_0(n) = n^{-1/2}T_0(n)$ with $T_0(n) \rightarrow \infty$.*

(1) *Fix a number $\epsilon > 0$, and suppose that $T_0(n) < (\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon}$. Then as $n \rightarrow \infty$ along S'_2 we have*

$$(1.15) \quad \mathcal{V}(X_{f_n, r}) \sim \frac{16}{3\pi} r^4 T^{-1}$$

uniformly for all

$$(1.16) \quad r_0 < r < n^{-1/2} (\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon}$$

and $f_n \in \mathcal{B}_n$, where $T := n^{1/2}r$.

(2) *Under the above notation let*

$$(1.17) \quad \hat{X}_{f_n, r} := \frac{X_{f_n, r} - \mathbb{E}[X_{f_n, r}]}{\sqrt{\mathcal{V}(X_{f_n, r})}}$$

be the standardized random L^2 -mass of f_n , $r_1 = r_1(n) = n^{-1/2}T_1(n)$, and suppose further that the sequence of numbers $T_1(n) > T_0(n)$ satisfies $T_1(n) = O(N^\xi)$ for every $\xi > 0$. Then for all $k \geq 3$ the k -th the moment of $\hat{X}_{f_n, r}$ converges, for $n \rightarrow \infty$ along S'_2 , to the standard Gaussian moment

$$(1.18) \quad \mathbb{E}[\hat{X}_{f_n, r}^k] \rightarrow \mathbb{E}[Z^k],$$

uniformly for $r_0 < r < r_1$ and $f_n \in \mathcal{B}_n$, where $Z \sim N(0, 1)$ is the standard Gaussian variable.

The claimed uniform asymptotics (1.15) of the variance means explicitly that, as $n \rightarrow \infty$ along S'_2 , one has

$$(1.19) \quad \sup_{\substack{r_0 < r < (\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon} \\ f_n \in \mathcal{B}_n}} \left| \frac{\mathcal{V}(X_{f_n, r})}{\frac{16}{3\pi} r^4 T^{-1}} - 1 \right| \rightarrow 0$$

and the uniform convergence (1.18) of the moments means that for every $k \geq 3$,

$$\sup_{\substack{r_0 < r < r_1 \\ f_n \in \mathcal{B}_n}} \left| \mathbb{E}[\hat{X}_{f_n, r}^k] - \mathbb{E}[Z^k] \right| \rightarrow 0.$$

Concerning the restricted range (1.16) in Theorem 1.1 (and (1.19)) for the possible radii, it is directly related to a well-known result on the angular distribution of lattice points in \mathcal{E}_n , for generic $n \in S_2$. Namely, it was shown [12] that \mathcal{E}_n , projected by homothety to the unit circle, is equidistributed, and moreover, a quantitative measure for the discrepancy is asserted (see section 2.1 below, and, in particular, (2.2)), satisfied by *generic* $n \in S_2$. Bourgain [7] observed that $f_n \in \mathcal{B}_n$, when averaged over $x \in \mathbb{T}^d$, exhibits Gaussianity in the following sense. Let $T > 0$ be a fixed number, and define the scaled function $\varphi_x : [-1, 1]^2 \rightarrow \mathbb{R}$ around x as

$$(1.20) \quad \varphi_x(y) := f_n \left(x + \frac{T}{\sqrt{n}} \cdot y \right),$$

i.e. the trace of f_n on the side- $2\frac{T}{\sqrt{n}}$ square centred at x . It was found [7], that, upon thinking of $x \in \mathbb{T}^2$ as *random*, and $\varphi_x(\cdot)$ as a *random field* indexed by $[-1, 1]^2$, it converges, in a suitable sense, to a particular *Gaussian field* (“monochromatic isotropic waves”) on \mathbb{R}^2 , restricted to $[-1, 1]^2$. This allows one to infer some results on the (deterministic) functions $f_n \in \mathcal{B}_n$ from the analogous results on the limit Gaussian random field. We may then reinterpret the quantitative version (2.2) of the angular equidistribution of lattice points as allowing the parameter T in (1.20) to grow as a (positive) logarithmic power of n , while still retaining the said asymptotic Gaussianity, also allowing for the comparison between the mass distribution of f_n w.r.t. the position and mass distribution of monochromatic isotropic waves. Our intuition regarding the possibility of carrying on the explained “de-randomisation” argument for establishing results of similar nature to the presented results was recently validated by Sartori [27].

An application of the standard theory [13, §XVI.3 Lemma 2] allows us to infer a uniform Central Limit Theorem for the random variables $\hat{X}_{f_n, r}$ from the convergence (1.18) of their respective moments to the Gaussian ones.

Corollary 1.2. *In the setting of Theorem 1.1 part (2), the distribution of the random variables $\{\hat{X}_{f_n, r}\}$ converges uniformly to the standard Gaussian distribution: as $n \rightarrow \infty$ along S'_2*

$$\text{meas}\{x \in \mathbb{T}^2 : \hat{X}_{f_n, r; x} \leq t\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-z^2/2} dz,$$

uniformly for $t \in \mathbb{R}$, $r_0 < r < r_1$ and $f_n \in \mathcal{B}_n$.

For the 3-dimensional case, for Bourgain’s eigenfunctions, we only claim precise asymptotic result on $\mathcal{V}(X_{f_n, r})$, the good news being that the claimed results are valid for *all* energies satisfying the natural congruence assumptions.

Theorem 1.3 (Asymptotics for the variance for $d = 3$, Bourgain’s eigenfunctions). *There exists a number $\eta > 0$ such that if $r_0 = r_0(n) = n^{-1/2}T_0(n)$ with $T_0(n) \rightarrow \infty$, then for all $n \not\equiv 0, 4, 7 \pmod{8}$ we have*

$$\mathcal{V}(X_{f_n, r}) \sim r^6 T^{-2},$$

uniformly for $r_0 < r < n^{-1/2+\eta}$ and $f_n \in \mathcal{B}_n$.

The meaning of the uniform statement in Theorem 1.3 is that

$$(1.21) \quad \sup_{\substack{r_0 < r < n^{-1/2+\eta} \\ f_n \in \mathcal{B}_n}} \left| \frac{\mathcal{V}(X_{f_n, r})}{r^6 T^{-2}} - 1 \right| \rightarrow 0$$

as $n \rightarrow \infty$ along $n \not\equiv 0, 4, 7 \pmod{8}$, cf. (1.19) in the 2-dimensional case.

1.5. Statement of the main results for $d = 2, 3$: more general upper and lower bounds. Let f_n be as in (1.5), and consider the vector

$$(1.22) \quad \underline{v} := (|c_\lambda|^2)_{\lambda \in \mathcal{E}_n} \in \mathbb{R}^{\mathcal{E}_n}$$

of the squared absolute values of its coefficients; we denote its normalised ℓ_∞ -norm

$$(1.23) \quad [\underline{v}]_\infty := N \cdot \max_{\lambda \in \mathcal{E}_n} |c_\lambda|^2.$$

Definition 1.4 (Ultraflat functions [15, Definition 1.9]). We say that an eigenfunction f_n in (1.5) is ϵ -ultraflat if its coefficients satisfy

$$(1.24) \quad [\underline{v}]_\infty \leq N^\epsilon.$$

Denote $\mathcal{U}_{n;\epsilon}$ to be the class of ϵ -ultraflat functions.

The following couple of theorems establish more general upper and lower bounds on $\mathcal{V}(X_{f_n,r})$ in the 2 and 3-dimensional cases respectively.

Theorem 1.5 (Bounds for the variance for ultra-flat eigenfunctions, $d = 2$). *There exists a density 1 sequence $S'_2 \subseteq S_2$ and an absolute constant $C > 0$ such that for every $\epsilon > 0$, $\eta > 0$, $r_0 = r_0(n) = n^{-1/2}T_0(n)$ with $T_0(n) \rightarrow \infty$ arbitrarily slowly, and $r = n^{-1/2}T > r_0$, as $n \rightarrow \infty$ along S'_2 we have*

$$(1.25) \quad T^{-1}N^{-2\epsilon} \ll \frac{\mathcal{V}(X_{f_n,r})}{r^4} \ll N^\epsilon \cdot \left(T^{-1} + (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \eta} \right)$$

uniformly for $r_0 < r < Cn^{-1/2}N^{1-\epsilon}$ and $f_n \in \mathcal{U}_{n;\epsilon}$, with the constant involved in the “ \ll ”-notation in (1.25) is absolute for the lower bound, and depends only on η for the upper bound. Moreover, the upper bound is valid for the extended range $r > r_0$ (with no upper bound on r imposed), and the lower bound is valid for every $n \in S_2$.

Theorem 1.6 (Bounds for the variance for ultra-flat eigenfunctions, $d = 3$). *There exists a number $\eta > 0$ and a constant $C > 0$ such that for every $\epsilon > 0$, $r_0 = r_0(n) = n^{-1/2}T_0(n)$ with $T_0(n) \rightarrow \infty$ arbitrarily slowly, $r = n^{-1/2}T > r_0$, and $n \not\equiv 0, 4, 7 \pmod{8}$ we have*

$$(1.26) \quad T^{-2}N^{-2\epsilon} \ll \frac{\mathcal{V}(X_{f_n,r})}{r^6} \ll N^\epsilon (T^{-2} + n^{-\eta}),$$

uniformly for $r_0 < r < Cn^{-1/2}N^{1-\epsilon}$ and $f_n \in \mathcal{U}_{n;\epsilon}$, where the constants involved in the “ \ll ”-notation are absolute. Moreover, the upper bound in (1.26) is valid for the extended range $r > r_0$.

For Bourgain’s eigenfunctions, the proofs of Theorem 1.5 and Theorem 1.6 yield slightly stronger bounds compared to (1.25) and (1.26), namely

$$T^{-1} \ll \frac{\mathcal{V}(X_{f_n,r})}{r^4} \ll T^{-1} + (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \epsilon}$$

for $d = 2$, and

$$T^{-2} \ll \frac{\mathcal{V}(X_{f_n,r})}{r^6} \ll T^{-2} + n^{-\eta}$$

for $d = 3$.

1.6. Outline of the paper. The rest of the paper is organised as follows. In section 2 we formulate Theorem 2.5, which, on one hand generalizes Theorem 1.1 for a larger class of flat eigenfunctions, and on the other hand, explicates a sufficient condition on $n \in S_2$ for its statements to hold; a few examples of application of Theorem 2.5, corresponding to different asymptotic behaviour of the variance (2.14), are also discussed. Section 4 is dedicated to giving a proof of the first part of Theorem 1.1 (resp. 1st part of Theorem 2.5), whereas the second part of Theorem 1.1 (resp. 2nd part of Theorem 2.5) is proved in section 5. Theorem 1.3, claiming the precise asymptotics for the L^2 -mass variance for Bourgain's eigenfunctions in 3d, is proved in section 6.

In section 7 we prove the various upper and lower bounds asserted by theorems 1.5 and 1.6. A refinement of Theorem 2.5, where rather than draw x w.r.t. the uniform measure on the full torus, x is drawn on balls slightly above Planck scale, is presented in section 8, and the additional subtleties of its proof as compared to the proof of Theorem 2.5 are highlighted. Finally, section 9 contains the proofs of all auxiliary lemmas, postponed in course of the proofs of the various results.

Acknowledgements. The authors of this manuscript wish to express their gratitude to J. Benatar, A. Granville, P. Kurlberg, Z. Rudnick, P. Sarnak and M. Sodin for numerous stimulating and fruitful discussions concerning various aspects of our work, and their interest in our research. It is a pleasure to thank the anonymous referee for his comments on an earlier version of this manuscript. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013), ERC grant agreement n° 335141.

2. ON THEOREM 1.1: CLT FOR MASS DISTRIBUTION, $d = 2$

In this section we focus on Theorem 1.1. Our first goal is to formulate a result, that on one hand generalises the statement of Theorem 1.1 to a larger class of eigenfunctions, and, on the other hand, provides a more explicit control over the generic numbers $n \in S_2$. To this end we discuss the angular distribution of $\lambda \in \mathcal{E}_n$ (section 2.1), and the spectral correlations (section 2.2), also used in the course of the proof of the 3-dimensional Theorem 1.3; we will be able to formulate Theorem 2.5, as prescribed above, by appealing to these. In section 2.4 we consider a few scenarios when Theorem 2.5 is applicable, prescribing different asymptotic behaviour for the variance (2.14).

2.1. Angular equidistribution of lattice points. For every $\lambda = (\lambda_1, \lambda_2) \in \mathcal{E}_n$, write $\lambda_1 + i\lambda_2 = \sqrt{n}e^{i\phi}$, and denote the various angles by

$$0 \leq \phi_1 < \phi_2 < \dots < \phi_N < 2\pi.$$

Recall that the discrepancy of the sequence ϕ_j is defined by

$$(2.1) \quad \Delta(n) = \sup_{0 \leq a \leq b \leq 2\pi} \left| \frac{1}{N} \cdot \#\{1 \leq j \leq N : \phi_j \in [a, b] \bmod 2\pi\} - \frac{(b-a)}{2\pi} \right|.$$

For every $\epsilon > 0$, we say that $n \in S_2$ satisfies the hypothesis $\mathcal{D}(n, \epsilon)$ if

$$(2.2) \quad \Delta(n) \leq (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \epsilon}.$$

By Erdős-Hall [12, Theorem 1], there exists a density one sequence $S'_2(\epsilon) \subseteq S_2$ such that $\mathcal{D}(n, \epsilon)$ is satisfied for every $n \in S'_2(\epsilon)$. By a standard diagonalization argument, there exists a density one sequence $S'_2 \subseteq S_2$ such that $\mathcal{D}(n, \epsilon)$ is satisfied for every $\epsilon > 0$ and $n \in S'_2$ sufficiently large. In particular, the angles $\{\phi_j\}$ are equidistributed mod 2π along this sequence, i.e., the lattice points are equidistributed on the corresponding circles.

2.2. Spectral correlations in $2d$ (and $3d$). For $d = 2$, while computing the moments of $X_{f_n, r}$ (e.g. for Bourgain's eigenfunction (1.14)), with x drawn uniformly on the whole of \mathbb{T}^2 , one exploits the orthogonality relations

$$\int_{\mathbb{T}^2} e(\langle \lambda, x \rangle) dx = \begin{cases} 0 & \lambda \neq 0 \\ 1 & \lambda = 0 \end{cases}$$

for $\lambda \in \mathbb{Z}^2$ to naturally encounter the length- l spectral correlation problem. That is, for $l \geq 2$ and $n \in S_2$ one is interested in the size of the length- l spectral correlation set

$$(2.3) \quad \mathcal{S}_n(l) = \left\{ (\lambda^1, \dots, \lambda^l) \in (\mathcal{E}_n)^l : \sum_{i=1}^l \lambda^i = 0 \right\},$$

which, by an elementary congruence obstruction argument modulo 2, is only non-empty for $l = 2k$ even.

In this case $l = 2k$ we further define the *diagonal* correlations set to be all the permutations of tuples of the form $(\lambda^1, -\lambda^1, \dots, \lambda^k, -\lambda^k)$:

$$(2.4) \quad \mathcal{D}_n(l) = \left\{ \pi(\lambda^1, -\lambda^1, \dots, \lambda^k, -\lambda^k) : \lambda^1, \dots, \lambda^k \in (\mathcal{E}_n)^k, \pi \in S_l \right\}.$$

The set \mathcal{D}_n is dominated by non-degenerate tuples (i.e. $\lambda^i \neq \pm \lambda^j$ for $i \neq j$), hence its size is asymptotic to

$$|\mathcal{D}_n(l)| = \frac{(2k)!}{2^k \cdot k!} N^k \cdot \left(1 + O_{N \rightarrow \infty} \left(\frac{1}{N} \right) \right).$$

Clearly, $\mathcal{D}_n(l) \subseteq \mathcal{S}_n(l)$ so that, in particular $\mathcal{S}_n(l) \gg N^{l/2}$. To the other end, we have $\mathcal{S}_n(2) = \mathcal{D}_n(2)$ by the definition, and both the precise statement

$$(2.5) \quad \mathcal{S}_n(4) = \mathcal{D}_n(4)$$

(used for the variance computation below) and the bound

$$|\mathcal{S}_n(l)| = O_{N \rightarrow \infty}(N^{l-2})$$

follow from Zygmund's elementary observation [31]. For $l = 6$, Bourgain (published in [23]) improved Zygmund's bound to

$$|\mathcal{S}_n(6)| = o_{N \rightarrow \infty}(N^4);$$

this was improved [6] to

$$|\mathcal{S}_n(6)| = O_{N \rightarrow \infty}(N^{7/2}),$$

valid for *all* $n \in S_2$.

If one is willing to excise a thin sequence in S_2 , then the more striking estimate [6]

$$|\mathcal{S}_n(6)| = |\mathcal{D}_n(6)| + O(N^{3-\gamma}),$$

with some $\gamma > 0$, is valid for a density 1 sequence $S'_2 \subseteq S_2$. More generally [7], for every $l \geq 6$ even, there exists a density 1 sequence $S'_2(l) \subseteq S_2$ and a number $\gamma_l > 0$ such that

$$(2.6) \quad |\mathcal{S}_n(l)| = |\mathcal{D}_n(l)| + O(N^{l/2-\gamma_l})$$

along $n \in S'_2(l)$. A standard diagonal argument then yields the existence of a density 1 sequence $S'_2 \subseteq S_2$ so that (2.6) is valid for *all* even $l \geq 4$.

Definition 2.1. Given an even number $l = 2k \geq 2$ we say that a sequence $S'_2 \subseteq S_2$ satisfies the length- l **diagonal domination** assumption if there exists a number $\gamma = \gamma_l > 0$ so that (2.6) holds.

For the 3-dimensional case under the consideration of Theorem 1.3 the analogous estimates to (2.6) are required to evaluate the relevant moments (1.12) of $X_{f_n, r}$. We define $\mathcal{S}_{3;n}$ and $\mathcal{D}_{3;n}$ analogously to (2.3) and (2.4) respectively, this time the λ^i are lying on the 2-sphere of radius \sqrt{n} . Unlike the lattice points lying on circles, Zygmund's argument is not applicable for the 2-sphere, so that an analogue of (2.5) is not valid; luckily the asymptotic statement

$$(2.7) \quad |\mathcal{S}_{3;n}(4)| = |\mathcal{D}_{3;n}(4)| + O\left(N^{7/4+\epsilon}\right),$$

a key input to the variance computation in Theorem 1.3, was recently established [3]. It was also shown in [3] that the asymptotic diagonal domination for the higher length correlations sets does not hold in the 3-dimensional case.

2.3. A more general version of Theorem 1.1, with explicit control over S'_2 . We are interested in extending Theorem 1.1 to a larger class of eigenfunctions. To this end, we introduce the following notation:

Notation 2.2. Let f_n be an eigenfunction on the 2-torus corresponding to coefficients $(c_\lambda)_{\lambda \in \mathcal{E}_n}$ via (1.5), and $\underline{v} \in \mathbb{R}^{\mathcal{E}_n} \simeq \mathbb{R}^N$ as above.

(1) Denote

$$(2.8) \quad A_4 = A_4(\underline{v}) = N \sum_{\lambda \in \mathcal{E}_n} |c_\lambda|^4 = N \cdot \|\underline{v}\|^2.$$

(2) Given $\lambda \in \mathcal{E}_n$ let λ_+ be the clockwise nearest neighbour of λ on $\sqrt{n}\mathcal{S}^1$, and

$$(2.9) \quad V(\underline{v}) := N \sum_{\lambda \in \mathcal{E}_n} \left| |c_{\lambda_+}|^2 - |c_\lambda|^2 \right|.$$

(3) Let

$$(2.10) \quad \tilde{V}(\underline{v}) = \frac{[\underline{v}]_\infty \cdot V(\underline{v})}{A_4(\underline{v})}.$$

The following Lemma, proved in section 9, summarizes some basic properties of the quantities in (1.23), (2.8), (2.9) and (2.10):

Lemma 2.3. *We have*

- (1) $1 \leq A_4 \leq [\underline{v}]_\infty$.
- (2) $[\underline{v}]_\infty \leq 1 + V(\underline{v})$.
- (3) $V(\underline{v}) \leq \tilde{V}(\underline{v}) \leq V(\underline{v}) (1 + V(\underline{v}))$.

By (1.8) we have that

$$(2.11) \quad A_4 = \cos(\theta)^{-2},$$

where $\theta = \theta_{f_n} = \theta(\underline{v}, \underline{v}_0)$ is the angle between \underline{v} and the vector $\underline{v}_0 = (\frac{1}{N})_{\lambda \in \mathcal{E}_n}$ corresponding to Bourgain's eigenfunctions, hence θ reflects the proximity of f_n to Bourgain's eigenfunction; by the first part of Lemma 2.3, the angle θ is restricted to the interval $\left[0, \arccos\left(1/\sqrt{N}\right)\right] \subseteq [0, \pi/2)$.

Definition 2.4 (Classes $\mathcal{F}_1(n; T(n), \eta(n))$ and $\mathcal{F}_2(n; T(n), \eta(n))$, $d = 2$). Given a sequence $T(n) \rightarrow \infty$ and a sequence $\eta(n) > 0$ we define:

- (1) A sequence $\{\mathcal{F}_1(n; T(n), \eta(n))\}_n$ of families of functions consisting for $n \in S_2$ of all functions f_n as in (1.5) satisfying

$$(2.12) \quad \mathcal{F}_1(n; T(n), \eta(n)) = \left\{ f_n : \tilde{V}(\underline{v}) < \eta(n) \cdot \frac{T(n)}{\log T(n)} \right\}.$$

- (2) A sequence $\{\mathcal{F}_2(n; T(n), \eta(n))\}_n$ of families of functions consisting for $n \in S_2$ of all functions f_n as in (1.5) satisfying

$$(2.13) \quad \mathcal{F}_2(n; T(n), \eta(n)) = \left\{ f_n : [\underline{v}]_\infty < T(n)^{\eta(n)} \right\},$$

where we recall the notation (1.23) for $[\underline{v}]_\infty$.

We are now in a position to state the generalized version of Theorem 1.1:

Theorem 2.5. *Let $r_0 = r_0(n) = n^{-1/2}T_0(n)$ with $T_0(n) \rightarrow \infty$, and $\eta(n) > 0$ any vanishing sequence $\eta(n) \rightarrow 0$.*

- (1) *Fix a number $\epsilon > 0$, and suppose that $T_0(n) < (\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon}$. Then, if $S'_2 \subseteq S_2$ is a sequence satisfying $\mathcal{D}(n, \epsilon/2)$ for all $n \in S'_2$, as $n \rightarrow \infty$ along S'_2 , we have*

$$(2.14) \quad \mathcal{V}(X_{f_n, r}) \sim \frac{16}{3\pi \cos^2 \theta_{f_n}} r^4 T^{-1}$$

with θ_{f_n} as in (2.11), uniformly for all $r_0 < r < n^{-1/2}(\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon}$ and $f_n \in \mathcal{F}_1(n; T(n), \eta(n))$, where $T := T(n) = n^{1/2}r$.

- (2) *Let $k \geq 3$ be an integer, $r_1 = r_1(n) = n^{-1/2}T_1(n)$, and suppose further that the sequence of numbers $T_1(n) > T_0(n)$ satisfies $T_1(n) = O(N^\xi)$ for every $\xi > 0$. Suppose that $S'_2 \subseteq S_2$ is a sequence satisfying the length- $2k$ diagonal domination assumption and the hypothesis $\mathcal{D}(n, \epsilon)$ for all $n \in S'_2$. Then the k -th the moment of $\hat{X}_{f_n, r}$ converges, as $n \rightarrow \infty$ along S'_2 , to the standard Gaussian moment*

$$\mathbb{E}[\hat{X}_{f_n, r}^k] \rightarrow \mathbb{E}[Z^k],$$

uniformly for $r_0 < r < r_1$ and $f_n \in \mathcal{F}_2(n; T(n), \eta(n))$ where $Z \sim N(0, 1)$ is the standard Gaussian variable.

Section 2.4 exhibits a few scenarios when Theorem 2.5 is applicable; as in these the true asymptotic behaviour of the variance (2.14) genuinely varies together with θ_{f_n} , this demonstrates that θ_{f_n} (and hence A_4) is the proper flatness measure of f_n , see also examples 2.7 and 2.8.

Corollary 2.6. *In the setting of Theorem 2.5 part (2), the distribution of the random variables $\{\hat{X}_{f_n, r}\}$ converges uniformly to the standard Gaussian distribution: as $n \rightarrow \infty$ along S'_2*

$$\text{meas}\{x \in \mathbb{T}^2 : \hat{X}_{f_n, r; x} \leq t\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-z^2/2} dz,$$

uniformly for $t \in \mathbb{R}$, $r_0 < r < r_1$, and $f_n \in \mathcal{F}_2(n; T(n), \eta(n))$.

2.4. Some examples of application of Theorem 2.5.

Example 2.7. *Let f_n be Bourgain's eigenfunction, so that $[\underline{v}]_\infty = A_4 = 1$ and $V(\underline{v}) = \tilde{V}(\underline{v}) = 0$. For every $\eta(n) > 0, T(n) > 1$ we have*

$$\mathcal{B}_n \subseteq \mathcal{F}_1(n; T(n), \eta(n)) \cap \mathcal{F}_2(n; T(n), \eta(n)).$$

Hence Theorem 2.5 implies Theorem 1.1.

The following example exhibits a scenario when an application of Theorem 2.5 yields a Central Limit Theorem for $X_{f_n, r}$, corresponding to asymptotic behaviour of the respective variance $\mathcal{V}(X_{f_n, r})$ which is very different from the behaviour in Theorem 1.1.

Example 2.8. Let $\epsilon > 0$, r_0 , and $T_0(n)$ as in Theorem 2.5, and $r_1 = r_1(n) = n^{-1/2}T_1(n) > r_0$ with $T_1(n) \leq (\log n)^{\frac{1}{2} \log \frac{\pi}{2} - \epsilon}$. There exists a density 1 sequence $S'_2 \subseteq S_2$ so that the following holds. Let $t = t(n) \in (0, 1)$ be a number satisfying $t(n) \gg \frac{1}{T_0(n)^\xi}$ for every $\xi > 0$, such that $N \cdot t$ is an integer. We choose an ordering $\lambda^1, \lambda^2, \dots, \lambda^N \in \mathcal{E}_n$ such that for every $1 \leq i \leq N - 1$ we have that λ^{i+1} is the (clockwise) nearest neighbour $\lambda^{i+1} = \lambda_+^i$, and set

$$(|c_{\lambda^1}|^2, \dots, |c_{\lambda^N}|^2) = (\underbrace{(Nt)^{-1}, \dots, (Nt)^{-1}}_{Nt \text{ times}}, 0, \dots, 0).$$

Then

$$(2.15) \quad \mathcal{V}(X_{f_n, r}) \sim \frac{16}{3\pi} r^4 t^{-1} T^{-1},$$

uniformly for $r_0 < r = n^{-1/2}T < r_1$, and f_n with coefficients c_λ as above. If, in addition, we have $T_1(n) = O(N^\xi)$ for every $\xi > 0$, then the distribution of the standardised random variable $\hat{X}_{f_n, r}$ converges to standard Gaussian uniformly.

Comparing (1.15) to (2.15) we observe that the asymptotic behaviour of the variance for the flat and the non-flat functions respectively is genuinely different, provided that we choose $t(n) \rightarrow 0$; we infer that the proposed flatness measure is the natural choice for this problem. One can also generalise Theorem 1.1 as follows:

Corollary 2.9. Let ϵ , r_0 , $T_0(n)$, r_1 and $T_1(n)$ be as in Theorem 2.5, and $g : S^1 \rightarrow \mathbb{R}$ a non-negative function of bounded variation such that $\|g\|_{L^1(S^1)} = 1$. For $n \in S_2$ and $\lambda \in \mathcal{E}_n$ we set $|\tilde{c}_\lambda|^2 := g(\lambda/\sqrt{n})$, and normalise the vector $\tilde{v} := (|\tilde{c}_\lambda|^2)_{\lambda \in \mathcal{E}_n}$ by setting $\underline{v} := \frac{\tilde{v}}{\|\tilde{v}\|_1}$, i.e.

$$(2.16) \quad v := (|c_\lambda|^2)_{\lambda \in \mathcal{E}_n} = \left(\frac{|\tilde{c}_\lambda|^2}{\sum_{\mu \in \mathcal{E}_n} |\tilde{c}_\mu|^2} \right)_{\lambda \in \mathcal{E}_n}.$$

Then along a generic sequence $S'_2 \subseteq S_2$ we have

$$\mathcal{V}(X_{f_n, r}) \sim \frac{16}{3\pi} \|g\|_{L^2(S^1)}^2 r^4 T^{-1},$$

uniformly for $r_0 < r = n^{-1/2}T < r_1$, and f_n with coefficients c_λ as in (2.16). If, in addition, we have $T_1(n) = O(N^\xi)$ for every $\xi > 0$, then the distribution of the standardised random variable $\hat{X}_{f_n, r}$ converges to standard Gaussian.

Proof. By Koksma's inequality (see e.g. [22]), $A_4(\underline{v}) \sim \|g\|_2^2$ along a density one sequence in S_2 . Also note that

$$V(\underline{v}) \ll V(g),$$

with the l.h.s. as in (2.9), and r.h.s. the variation of g on S^1 . In light of Lemma 2.3, both parts of Corollary 2.9 follow from Theorem 2.5. \square

3. NOTATION

For the convenience of the reader, we summarize here the notation used in our paper.

$S_d = \{n = a_1^2 + \dots + a_d^2 : a_1, \dots, a_d \in \mathbb{Z}\}$: the set of integers expressible as a sum of d squares, see (1.6).

$\mathcal{E}_n = \mathcal{E}_{d;n} = \{\lambda \in \mathbb{Z}^d : \|\lambda\|^2 = n\}$: the standard lattice points lying on the $(d-1)$ -dimensional sphere (a circle for $d=2$) of radius- \sqrt{n} , see (1.7).

$f_n(x) = \sum_{\lambda \in \mathcal{E}_n} c_\lambda e(\langle x, \lambda \rangle)$: the toral Laplace eigenfunctions, see (1.5).

$N = N_{d;n} = \#\mathcal{E}_n$: the number of lattice points on the $(d-1)$ -dimensional sphere (a circle for $d=2$) of radius- \sqrt{n} , see (1.9).

$B_x(r)$: the radius r geodesic ball in \mathbb{T}^d centred at x .

$X_{f_n,r} = X_{f_n,r;x} = \int_{B_x(r)} f_n(y)^2 dy$: the L^2 -mass of f_n restricted to $B_x(r)$, where x is drawn randomly uniformly in \mathbb{T}^d , see (1.10).

$\mathbb{E}[X_{f_n,r}] = \int_{\mathbb{T}^d} X_{f_n,r;x} dx$: the expected value of $X_{f_n,r}$, see (1.11).

$\mathcal{V}(X_{f_n,r}) = \mathbb{E}[(X_{f_n,r} - \mathbb{E}[X_{f_n,r}])^2]$: the variance of $X_{f_n,r}$, see (1.13).

$\hat{X}_{f_n,r} := \frac{X_{f_n,r} - \mathbb{E}[X_{f_n,r}]}{\sqrt{\mathcal{V}(X_{f_n,r})}}$: the standardized random L^2 -mass of f_n , see (1.17).

$T = n^{1/2}r$.

$\underline{v} = (|c_\lambda|^2)_{\lambda \in \mathcal{E}_n} \in \mathbb{R}^{\mathcal{E}_n}$: the vector of the squared absolute values of the coefficients of f_n , see (1.22).

$[v]_\infty = N \cdot \max_{\lambda \in \mathcal{E}_n} |c_\lambda|^2$: the normalised ℓ_∞ -norm of \underline{v} , see (1.23).

\mathcal{B}_n : the class of Bourgain's eigenfunctions $f_n(x) = \frac{1}{\sqrt{N}} \sum_{\lambda \in \mathcal{E}_n} \varepsilon_\lambda e(\langle x, \lambda \rangle)$, where $\varepsilon_\lambda = \pm 1$ for every $\lambda \in \mathcal{E}_n$, see (1.14).

$\mathcal{U}_{n;\epsilon}$: the class of ϵ -ultraflat functions, where $[v]_\infty \leq N^\epsilon$, see (1.24).

$A_4 = A_4(\underline{v}) = N \sum_{\lambda \in \mathcal{E}_n} |c_\lambda|^4 = N \cdot \|\underline{v}\|^2$, see (2.8).

$\theta = \theta_{f_n} = \theta(\underline{v}, \underline{v}_0)$: the angle between \underline{v} and the vector $\underline{v}_0 = (\frac{1}{N})_{\lambda \in \mathcal{E}_n}$ corresponding to Bourgain's eigenfunctions, see (2.11).

$V(\underline{v}) = N \sum_{\lambda \in \mathcal{E}_n} \left| |c_{\lambda_+}|^2 - |c_\lambda|^2 \right|$, where λ_+ is the clockwise nearest neighbour of λ on $\sqrt{n}\mathcal{S}^1$, see (2.9).

$\tilde{V}(\underline{v}) = \frac{[v]_\infty \cdot V(\underline{v})}{A_4(\underline{v})}$, see (2.10).

$\mathcal{F}_1(n; T(n), \eta(n)) = \left\{ f_n : \tilde{V}(\underline{v}) < \eta(n) \cdot \frac{T(n)}{\log T(n)} \right\}$, see (2.12).

$\mathcal{F}_2(n; T(n), \eta(n)) = \left\{ f_n : [v]_\infty < T(n)^{\eta(n)} \right\}$, see (2.13).

$\hat{\lambda} = \lambda/\sqrt{n}$: the projection of $\lambda \in \mathcal{E}_n$ onto \mathcal{S}^{d-1} .

$\Delta(n) = \sup_{0 \leq a \leq b \leq 2\pi} \left| \frac{1}{N} \cdot \# \{1 \leq j \leq N : \phi_j \in [a, b] \bmod 2\pi\} - \frac{(b-a)}{2\pi} \right|$: the discrepancy of the angles ϕ_j corresponding to the lattice points $\mathcal{E}_{2;n}$, see (2.1).

Hypothesis $\mathcal{D}(n, \epsilon)$ holds if $\Delta(n) \leq (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \epsilon}$, see (2.2).

$\Delta_3(n) = \sup_{\substack{x \in \mathcal{S}^2 \\ 0 < r \leq 2}} \left| \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_{3;n} : \left| \widehat{\lambda} - x \right| \leq r \right\} - \frac{r^2}{4} \right|$: the spherical cap discrepancy of the points $\mathcal{E}_{3;n}$, see (6.2).

$\mathcal{S}_n(l) = \left\{ (\lambda^1, \dots, \lambda^l) \in (\mathcal{E}_n)^l : \sum_{i=1}^l \lambda^i = 0 \right\}$: the length- l spectral correlation set, see (2.3).

$\mathcal{D}_n(l) = \left\{ \pi(\lambda^1, -\lambda^1, \dots, \lambda^k, -\lambda^k) : \lambda^1, \dots, \lambda^k \in (\mathcal{E}_n)^k, \pi \in S_l \right\}$: the diagonal correlations set, see (2.4).

$\mathcal{A}_n(2k) = \{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{D}_n(2k) : \forall 1 \leq i \leq k \lambda_{2i-1} \neq -\lambda_{2i}\}$: the set of “admissible” $2k$ -tuples of lattice points, see (5.1).

$S(\lambda_1, \dots, \lambda_{2k})$: the structure set of an admissible $2k$ -tuple $(\lambda_1, \dots, \lambda_{2k})$, see (5.1).

$J_\alpha(x)$: the Bessel function of the first kind of order α .

$g_d(x) = \frac{J_{d/2}(2\pi x)}{(2\pi x)^{d/2}}$: the Fourier transform of the characteristic function of the unit ball in \mathbb{R}^d , see (4.4).

$h_2(x) = \frac{J_1(2\pi x)^2}{(2\pi x)^2}$, see (4.6).

$h_3(x) = 2\pi^{-1}(2\pi x)^{-4} \left(\frac{\sin 2\pi x}{2\pi x} - \cos 2\pi x \right)^2$, see (4.8).

$F_{\lambda_0}(s) = \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_{2;n} : \left\| \widehat{\lambda} - \widehat{\lambda}_0 \right\| \leq s \right\}$, see (4.9).

$F(s) = F_{f_n}(s) = \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_{2;n} \\ 0 < \left\| \widehat{\lambda} - \widehat{\lambda}' \right\| \leq s}} |c_\lambda|^2 |c_{\lambda'}|^2$, see (4.10).

$F_3(s) = \frac{1}{N^2} \cdot \# \left\{ \lambda \neq \lambda' \in \mathcal{E}_{3;n} : \left\| \widehat{\lambda} - \widehat{\lambda}' \right\| \leq s \right\}$, see (6.1).

$\mathbb{E}_{x_0, \rho}[X_{f_n, r}] = \frac{1}{\text{Vol}(B_{x_0}(\rho))} \int_{B_{x_0}(\rho)} X_{f_n, r; x} dx$: the “restricted” expected value of $X_{f_n, r}$, see (8.1).

$\mathcal{V}_{x_0, \rho}(X_{f_n, r}) = \mathbb{E}_{x_0, \rho}[(X_{f_n, r} - \mathbb{E}_{x_0, \rho}[X_{f_n, r}])^2]$: the restricted variance of $X_{f_n, r}$, see (8.3).

$\mathcal{C}_n(l; K) = \left\{ (\lambda^1, \dots, \lambda^l) \in \mathcal{E}_n^l : 0 < \left\| \sum_{j=1}^l \lambda^j \right\| \leq K \right\}$: the set of length- l spectral quasi-correlations, see (8.5).

Hypothesis $\mathcal{A}(n; l, \delta)$ holds if $\mathcal{C}_n(l; n^{1/2-\delta}) = \emptyset$, see (8.6).

4. PROOF OF THEOREM 2.5, PART 1: ASYMPTOTICS FOR THE VARIANCE, $d = 2$.

4.1. Expressing the variance. We begin with some preliminary expressions for the variance. Note that if x is drawn randomly, uniformly on \mathbb{T}^d , then

$$(4.1) \quad \mathbb{E}[X_{f_n, r}] = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d,$$

and therefore in this case, we have

$$(4.2) \quad \mathcal{V}(X_{f_n, r}) = \int_{\mathbb{T}^d} \left(\int_{B_x(r)} f_n(y)^2 dy - \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d \right)^2 dx.$$

Let $J_\alpha(x)$ be the Bessel function of the first kind of order α . The following lemma, proved in section 9, explicates the inner integral in (4.2):

Lemma 4.1. *We have*

$$(4.3) \quad \int_{B_x(r)} f_n(y)^2 dy - \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d = (2\pi)^{d/2} r^d \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} c_\lambda \overline{c_{\lambda'}} e(\langle x, \lambda - \lambda' \rangle) g_d(r \|\lambda - \lambda'\|),$$

where

$$(4.4) \quad g_d(x) := \frac{J_{d/2}(2\pi x)}{(2\pi x)^{d/2}}$$

is the Fourier transform of the characteristic function of the unit ball in \mathbb{R}^d .

The following formula for the variance follows from Lemma 4.1, (2.5) and (2.7):

Lemma 4.2.

(1) (Granville-Wigman [15, Lemma 2.1]) For $d = 2$ we have

$$(4.5) \quad \mathcal{V}(X_{f_n, r}) = 8\pi^2 r^4 \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} |c_\lambda|^2 |c_{\lambda'}|^2 h_2(r \|\lambda - \lambda'\|)$$

where

$$(4.6) \quad h_2(x) := \frac{J_1(2\pi x)^2}{(2\pi x)^2}.$$

(2) For $d = 3$ and for every $\epsilon > 0$, we have

$$(4.7) \quad \begin{aligned} \mathcal{V}(X_{f_n, r}) &= 16\pi^3 r^6 \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} |c_\lambda|^2 |c_{\lambda'}|^2 h_3(r \|\lambda - \lambda'\|) \\ &\quad + O([v]_\infty^2 r^6 N^{-1/4+\epsilon}), \end{aligned}$$

where

$$(4.8) \quad h_3(x) := 2\pi^{-1} (2\pi x)^{-4} \left(\frac{\sin 2\pi x}{2\pi x} - \cos 2\pi x \right)^2.$$

Note that functions g_2 and h_2 satisfy the following properties:

Lemma 4.3 ([14, (6.575.2), (8.440), (8.451.1), (8.472.2)]). *We have*

- (1) $\int_0^\infty h_2(s) ds = \frac{2}{3\pi^2}.$
- (2) $g_2(s) \sim \frac{1}{2} \quad (s \rightarrow 0).$

$$(3) \quad g_2(s) \ll s^{-3/2} \quad (s \rightarrow \infty).$$

$$(4) \quad g_2'(s) = -\frac{J_2(2\pi s)}{s} \ll (1+s)^{-3/2}.$$

4.2. Proof of Theorem 2.5, part 1:

Definition 4.4. For $\lambda \in \mathcal{E}_n$ let $\widehat{\lambda} = \lambda/\sqrt{n}$ be the projection of λ onto the unit circle \mathcal{S}^1 .

(1) For $\lambda_0 \in \mathcal{E}_n$ and $0 \leq s \leq 2$, denote

$$(4.9) \quad F_{\lambda_0}(s) = \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_n : \left\| \widehat{\lambda} - \widehat{\lambda_0} \right\| \leq s \right\}.$$

(2) For $0 \leq s \leq 2$ denote

$$(4.10) \quad F(s) = F_{f_n}(s) = \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ 0 < \left\| \widehat{\lambda} - \widehat{\lambda'} \right\| \leq s}} |c_\lambda|^2 |c_{\lambda'}|^2.$$

Recall that $\widetilde{V}(\underline{v}) = \cos^2 \theta \cdot [\underline{v}]_\infty V(\underline{v})$ by (2.10) and (2.11).

Proposition 4.5. *We have*

$$F(s) = \frac{s}{\pi \cos^2 \theta} \left(1 + O \left(s^2 + s^{-1} \Delta(n) + \widetilde{V}(\underline{v})s + \widetilde{V}(\underline{v})s^{-1} \Delta(n)^2 \right) \right).$$

We postpone the proof of Proposition 4.5 until section 4.3 to present the proof of the first part of Theorem 2.5 (that yields the first part of Theorem 1.1):

Proof of Theorem 2.5, part 1 assuming Proposition 4.5. Assume that $n \in S_2$ satisfies the hypothesis $\mathcal{D}(n, \epsilon/2)$. We may rewrite (4.5) as

$$(4.11) \quad \mathcal{V}(X_{f_n, r}) = 8\pi^2 r^4 \int_0^2 h_2(Ts) \, dF(s).$$

We apply integration by parts to (4.11) twice, in opposite directions: first, by integration by parts and Proposition 4.5, we get

$$(4.12) \quad \begin{aligned} 8\pi^2 r^4 \int_0^2 h_2(Ts) \, dF(s) &= 8\pi^2 r^4 h_2(2T) F(2) - 8\pi^2 r^4 \int_0^2 F(s) \, dh_2(Ts) \\ &= 8\pi^2 r^4 h_2(2T) F(2) - 8\pi r^4 \cos^{-2} \theta \int_0^2 s \, dh_2(Ts) \\ &\quad + Err(X_{f_n, r}) \end{aligned}$$

where

$$Err(X_{f_n, r}) \ll r^4 \cos^{-2} \theta \int_0^2 \left(s^3 + \Delta(n) + \widetilde{V}(\underline{v})s^2 + \widetilde{V}(\underline{v})\Delta(n)^2 \right) T |h_2'(Ts)| \, ds.$$

Integrating by parts again, the first two terms on the r.h.s of (4.12) satisfy

$$(4.13) \quad \begin{aligned} 8\pi^2 r^4 h_2(2T) F(2) - 8\pi r^4 \cos^{-2} \theta \int_0^2 s \, dh_2(Ts) &= 8\pi^2 r^4 h_2(2T) F(2) \\ &\quad - 16\pi r^4 h_2(2T) \cos^{-2} \theta + 8\pi r^4 \cos^{-2} \theta \int_0^2 h_2(Ts) \, ds. \end{aligned}$$

By the first and the third parts of Lemma 4.3,

$$(4.14) \quad \int_0^2 h_2(Ts) \, ds = \frac{1}{T} \int_0^{2T} h_2(s) \, ds = \frac{2}{3\pi^2} T^{-1} + O(T^{-3}),$$

and therefore, substituting (4.14) into (4.13), we obtain

(4.15)

$$8\pi^2 r^4 h_2(2T) F(2) - 8\pi r^4 \cos^{-2} \theta \int_0^2 s \, dh_2(Ts) = \frac{16}{3\pi} \cos^{-2} \theta r^4 T^{-1} + O(\cos^{-2} \theta r^4 T^{-3}).$$

By the fourth part of Lemma 4.3,

$$\begin{aligned} \int_0^2 T |h'_2(Ts)| \, ds &= \int_0^{2T} |h'_2(s)| \, ds \leq \int_0^\infty |h'_2(s)| \, ds < \infty, \\ \int_0^2 s^2 T |h'_2(Ts)| \, ds &= T^{-2} \int_0^{2T} s^2 |h'_2(s)| \, ds \ll T^{-2} \log T \end{aligned}$$

and

$$\int_0^2 s^3 T |h'_2(Ts)| \, ds = T^{-3} \int_0^{2T} s^3 |h'_2(s)| \, ds \ll T^{-2},$$

and therefore for n satisfying $\mathcal{D}(n, \epsilon/2)$,

$$\begin{aligned} \text{Err}(X_{f_n, r}) &\ll \cos^{-2} \theta r^4 \left(T^{-2} + \Delta(n) + \tilde{V}(\underline{v}) T^{-2} \log T + \tilde{V}(\underline{v}) \Delta(n)^2 \right) \\ (4.16) \quad &\ll \cos^{-2} \theta r^4 \left(T^{-2} + (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \frac{\epsilon}{2}} + \tilde{V}(\underline{v}) T^{-2} \log T + \tilde{V}(\underline{v}) (\log n)^{-\log \frac{\pi}{2} + \epsilon} \right), \end{aligned}$$

and (2.14) follows from (4.12), (4.15) and (4.16). \square

Note that by (4.16), for Bourgain's eigenfunctions, for almost all $n \in S_2$ we have

$$\sup_{\substack{r > r_0 \\ f_n \in \mathcal{B}n}} \left| \frac{\mathcal{V}(X_{f_n, r})}{r^4} - \frac{16}{3\pi} T^{-1} \right| = O\left(T_0^{-2} + (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \epsilon}\right)$$

for every $\epsilon > 0$, and in particular

$$(4.17) \quad \mathcal{V}(X_{f_n, r}) = o(r^4)$$

uniformly for $r > r_0$ for a density one sequence in S_2 . Therefore, (4.17) serves as a refinement of [15, Corollary 1.10] for this specific case (for a density one sequence in S_2), since [15, Corollary 1.10] yields $\mathcal{V}(X_{f_n, r}) = o(r^4)$ under the additional assumption $T_0 \gg n^{4\epsilon}$.

4.3. Proof of Proposition 4.5. In this section we prove Proposition 4.5. First, we define a binary relation on \mathcal{E}_n :

Definition 4.6. for $\lambda \neq -\lambda' \in \mathcal{E}_n$, we say that $\lambda \prec \lambda'$ if the arc on the circle $\sqrt{n}S^1$ that connects λ to λ' counter-clockwise to λ' is shorter than the arc that connects them clockwise to λ' . Recall that λ_+ is the clockwise nearest neighbour of λ on $\sqrt{n}S^1$. The proof of Proposition 4.5 employs the following auxiliary lemma to be proved at section 9, establishing (4.10) in the particular case $|c_\lambda|^2 = 1$ for every $\lambda \in \mathcal{E}_n$:

Lemma 4.7. Fix $\lambda' \in \mathcal{E}_n$. For $0 \leq s < 2$, we have

$$(4.18) \quad \frac{1}{N} \cdot \#\left\{ \lambda \in \mathcal{E}_n : \lambda \succeq \lambda', \left\| \widehat{\lambda} - \widehat{\lambda}' \right\| \leq s \right\} = \frac{s}{2\pi} + O(s^3 + \Delta(n))$$

where the constant involved in the 'O'-notation in (4.18) is absolute.

Remark 4.8. The estimate (4.18) is also valid with either ' \succ ', ' \preceq ' or ' \prec ' in place of ' \succeq '.

We are now in a position to prove Proposition 4.5:

Proof of Proposition 4.5 assuming Lemma 4.7. First, we write

$$(4.19) \quad F(s) = \sum_{\lambda' \in \mathcal{E}_n} |c_{\lambda'}|^2 \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \preceq \lambda'}} |c_{\lambda}|^2 + \sum_{\lambda' \in \mathcal{E}_n} |c_{\lambda'}|^2 \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \succeq \lambda'}} |c_{\lambda}|^2 + O\left(\frac{A_4}{N}\right).$$

Using summation by parts, we get that for every $\lambda' \in \mathcal{E}_n$

$$(4.20) \quad \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \preceq \lambda'}} |c_{\lambda}|^2 = |c_{\lambda'}|^2 \cdot \#\left\{\lambda \in \mathcal{E}_n : \lambda \preceq \lambda', \|\hat{\lambda} - \hat{\lambda}'\| \leq s\right\} \\ - \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \prec \lambda'}} \left(|c_{\lambda_+}|^2 - |c_{\lambda}|^2\right) \cdot \#\left\{\mu \in \mathcal{E}_n : \mu \preceq \lambda, \|\hat{\mu} - \hat{\lambda}'\| \leq s\right\}.$$

By Lemma 4.7, the contribution of the first term on the r.h.s of (4.20) to $F(s)$ is

$$(4.21) \quad \sum_{\lambda' \in \mathcal{E}_n} |c_{\lambda'}|^4 \cdot \#\left\{\lambda \in \mathcal{E}_n : \lambda \preceq \lambda', \|\hat{\lambda} - \hat{\lambda}'\| \leq s\right\} = A_4 \cdot (s/2\pi + O(s^3 + \Delta(n))).$$

The contribution of the sum on the r.h.s of (4.20) to $F(s)$ is

$$(4.22) \quad \sum_{\lambda' \in \mathcal{E}_n} |c_{\lambda'}|^2 \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \prec \lambda'}} \left(|c_{\lambda_+}|^2 - |c_{\lambda}|^2\right) \cdot \#\left\{\mu \in \mathcal{E}_n : \mu \preceq \lambda, \|\hat{\mu} - \hat{\lambda}'\| \leq s\right\} \\ \ll N(s + \Delta(n)) \sum_{\lambda \in \mathcal{E}_n} \left||c_{\lambda_+}|^2 - |c_{\lambda}|^2\right| \sum_{\substack{\lambda' \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \prec \lambda'}} |c_{\lambda'}|^2 \ll (s + \Delta(n))^2 [\underline{v}]_{\infty} V(\underline{v}).$$

By (4.21) and (4.22), we have

$$(4.23) \quad \sum_{\substack{\lambda \in \mathcal{E}_n \\ \|\hat{\lambda} - \hat{\lambda}'\| \leq s \\ \lambda \preceq \lambda'}} |c_{\lambda}|^2 = A_4 \cdot (s/2\pi + O(s^3 + \Delta(n))) + O\left((s + \Delta(n))^2 [\underline{v}]_{\infty} V(\underline{v})\right).$$

By symmetry, the second summation in (4.19) obeys (4.20) with ' \succ ', ' \succeq ' and $|c_{\lambda_-}|^2$ in place of ' \prec ', ' \preceq ' and $|c_{\lambda_+}|^2$, where λ_- is the counter-clockwise nearest neighbour to λ . The statement of Proposition 4.5 follows from the analogues of the estimates (4.21), (4.22) and (4.23). \square

5. PROOF OF THEOREM 2.5, PART 2: GAUSSIAN MOMENTS, $d = 2$.

In this section we study the higher moments of $\hat{X}_{f_n, r}$ defined in (1.17), and prove the second part of Theorem 2.5, also implying the second part of Theorem 1.1.

The proof of the following lower bound for $\mathcal{V}(X_{f_n, r})$ with $f_n \in \mathcal{F}_2(n; T(n), \eta(n))$ goes along the same lines as the proof of the lower bound in Theorem 1.5 below:

Lemma 5.1. *In the setting of Theorem 2.5 part (2), we have*

$$\frac{\mathcal{V}(X_{f_n, r})}{r^4} \gg T(n)^{-1-2\eta(n)}$$

uniformly for $r_0 < r < r_1$ and $f_n \in \mathcal{F}_2(n; T(n), \eta(n))$.

Before proceeding to the proof of Theorem 2.5, we introduce some notation:

Definition 5.2.

(1) Define the set of “admissible” $2k$ -tuples of lattice points by

$$(5.1) \quad \mathcal{A}_n(2k) = \{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{D}_n(2k) : \forall 1 \leq i \leq k \ \lambda_{2i-1} \neq -\lambda_{2i}\}.$$

(2) Given an admissible $2k$ -tuple of lattice points $(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{A}_n(2k)$, let \sim be the equivalence relation on the set $\{1, \dots, 2k\}$, generated by:

- (a) $2i-1 \sim 2i$ for every $1 \leq i \leq k$.
- (b) $j \sim j'$ if $\lambda_j + \lambda_{j'} = 0$.

Let $\{\Lambda_1, \dots, \Lambda_m\}$ be the partition of $\{1, \dots, 2k\}$ into equivalence classes of \sim , and denote $l_j = \#\Lambda_m/2$ for $1 \leq j \leq m$, so that $\sum_{j=1}^m l_j = k$; clearly, $2 \leq l_j \in \mathbb{Z}$ for every $1 \leq j \leq m$. We call the multiset

$$(5.2) \quad S(\lambda_1, \dots, \lambda_{2k}) := \{l_1, \dots, l_m\}$$

the structure set of the $2k$ -tuple $(\lambda_1, \dots, \lambda_{2k})$.

Recall that the moments of a standard Gaussian random variable $Z \sim N(0, 1)$ are

$$\mathbb{E}[Z^k] = \begin{cases} (k-1)!! & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

We are now in a position to prove the second part of Theorem 2.5.

Proof of Theorem 2.5, part 2. By the length- $2k$ diagonal domination assumption, we have

$$(5.3) \quad \mathbb{E}[\hat{X}_{f_n, r}^k] = (2\pi)^k r^{2k} \mathcal{V}(X_{f_n, r})^{-k/2} \sum_{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{A}_n(2k)} \prod_{j=1}^k c_{\lambda_{2j-1}} c_{\lambda_{2j}} g_2(r \|\lambda_{2j-1} + \lambda_{2j}\|) \\ + O\left(\mathcal{V}(X_{f_n, r})^{-k/2} [v]_\infty^k r^{2k} N^{-\gamma}\right)$$

for some $\gamma > 0$. We can rearrange the summation in (5.3), first summing over all possible structure sets $\mathcal{L} = \{l_1, \dots, l_m\}$ and then summing over the admissible $2k$ -tuples $(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{E}_n^{2k}$ with the given structure set $S(\lambda_1, \dots, \lambda_{2k}) = \mathcal{L}$: let

$$S_{\mathcal{L}} := \sum_{\substack{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{A}_n(2k) \\ S(\lambda_1, \dots, \lambda_{2k}) = \mathcal{L}}} \prod_{j=1}^k c_{\lambda_{2j-1}} c_{\lambda_{2j}} g_2(r \|\lambda_{2j-1} + \lambda_{2j}\|),$$

so that we may rewrite the summation on the r.h.s. of (5.3) as

$$(5.4) \quad \sum_{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{A}_n(2k)} \prod_{j=1}^k c_{\lambda_{2j-1}} c_{\lambda_{2j}} g_2(r \|\lambda_{2j-1} + \lambda_{2j}\|) = \sum_{\substack{l_1 + \dots + l_m = k \\ l_1, \dots, l_m \geq 2}} \sum_{\substack{(\lambda_1, \dots, \lambda_{2k}) \in \mathcal{A}_n(2k) \\ S(\lambda_1, \dots, \lambda_{2k}) = \mathcal{L}}} S_{\mathcal{L}}.$$

For a fixed structure set $\mathcal{L} = \{l_1, \dots, l_m\}$, we have

$$(5.5) \quad S_{\mathcal{L}} = a(\mathcal{L}) \prod_{j=1}^m \sum_{\lambda_1, \dots, \lambda_{l_j} \in \mathcal{E}_n} |c_{\lambda_1}|^2 g_2(r \|\lambda_{l_j} - \lambda_1\|) \prod_{i=1}^{l_j-1} |c_{\lambda_{i+1}}|^2 g_2(r \|\lambda_i - \lambda_{i+1}\|) + O\left([v]_\infty^k N^{-1}\right)$$

where $a(\mathcal{L})$ is a constant depending on \mathcal{L} ; omitting the condition that the lattice points are distinct on the r.h.s of (5.5) is absorbed within the error term in (5.5). Thus,

(5.6)

$$S_{\mathcal{L}} \ll [v]_{\infty}^k N^{-k} \prod_{j=1}^m \sum_{\lambda_1 \in \mathcal{E}_n} \sum_{\lambda_2 \in \mathcal{E}_n} |g_2(r \|\lambda_2 - \lambda_1\|)| \cdots \sum_{\lambda_{l_j} \in \mathcal{E}_n} |g_2(r \|\lambda_{l_j-1} - \lambda_{l_j}\|)| + [v]_{\infty}^k N^{-1}.$$

Recall the definition of F_{λ_0} in (4.9). By Lemma 4.7, we have

$$(5.7) \quad F_{\lambda_0}(s) = \frac{s}{\pi} + O(s^3 + \Delta(n)) = O(s + \Delta(n)).$$

Thus, by Lemma 4.3 and (5.7), we have that

$$(5.8) \quad \begin{aligned} \frac{1}{N} \sum_{\lambda \in \mathcal{E}_n} |g_2(r \|\lambda - \lambda_0\|)| &= \int_0^2 |g_2(Ts)| dF_{\lambda_0}(s) \\ &= |g_2(2T)| - \frac{1}{2N} + O\left(\int_0^2 (s + \Delta(n)) T |g_2'(Ts)| ds\right) \\ &= O\left(T^{-3/2} + (\Delta(n) + T^{-1}) \int_0^{2T} |g_2'(s)| ds\right) \\ &= O(T^{-1}) \end{aligned}$$

for n satisfying the hypothesis $\mathcal{D}(n, \epsilon)$. Applying (5.8) to each of the $l_j - 1$ inner summations in (5.6), we obtain

$$S_{\mathcal{L}} \ll [v]_{\infty}^k N^{-k+m} \prod_{j=1}^m (NT^{-1})^{l_j-1} + [v]_{\infty}^k N^{-1} \ll [v]_{\infty}^k T^{-k+m}.$$

Let $\mathcal{L}_0 = \{2, 2, \dots, 2, \}$. Note that if $\mathcal{L} \neq \mathcal{L}_0$ then $m \leq \frac{k-1}{2}$ and therefore

$$(5.9) \quad S_{\mathcal{L}} = O\left([v]_{\infty}^k T^{-\frac{k+1}{2}}\right).$$

If $\mathcal{L} = \mathcal{L}_0$ (this is a viable option for k even), then

$$(5.10) \quad S_{\mathcal{L}_0} = 2^{k/2} (k-1)!! \left[\sum_{\lambda_1 \neq \lambda_2 \in \mathcal{E}_n} |c_{\lambda_1}|^2 |c_{\lambda_2}|^2 h_2(r \|\lambda_1 - \lambda_2\|) \right]^{k/2} + O\left([v]_{\infty}^k N^{-1}\right).$$

By (4.5),

$$(5.11) \quad \sum_{\lambda_1 \neq \lambda_2 \in \mathcal{E}_n} |c_{\lambda_1}|^2 |c_{\lambda_2}|^2 h_2(r \|\lambda_1 - \lambda_2\|) = \frac{\mathcal{V}(X_{f_n, r})}{8\pi^2 r^4}.$$

Hence, (5.10) and (5.11) yield

$$(5.12) \quad S_{\mathcal{L}_0} = (k-1)!! \left(\frac{\mathcal{V}(X_{f_n, r})}{4\pi^2 r^4} \right)^{k/2} + O\left([v]_{\infty}^k N^{-1}\right).$$

Substituting (5.9) and (5.12) into (5.4) and applying Lemma 5.1, we finally obtain that for k even

$$\left| \mathbb{E}[\hat{X}_{f_n, r}^k] - (k-1)!! \right| \ll T^{k\eta(n)} [v]_{\infty}^k \left(T^{-1/2} + T^{k/2} N^{-\min\{1, \gamma\}} \right) \ll T^{-1/2+2k\eta(n)}$$

and since for k odd $\mathcal{L} = \mathcal{L}_0$ is not a viable option, we obtain

$$\mathbb{E}[\hat{X}_{f_n, r}^k] \ll T^{k\eta(n)} [v]_{\infty}^k \left(T^{-1/2} + T^{k/2} N^{-\min\{1, \gamma\}} \right) \ll T^{-1/2+2k\eta(n)},$$

and the second part of Theorem 2.5 follows. \square

6. PROOF OF THEOREM 1.3: ASYMPTOTICS FOR THE VARIANCE, $d = 3$

6.1. Proof of Theorem 1.3. Denote

$$(6.1) \quad F_3(s) = \frac{1}{N^2} \cdot \# \left\{ \lambda \neq \lambda' \in \mathcal{E}_n : \|\hat{\lambda} - \hat{\lambda}'\| \leq s \right\}$$

(cf. (4.10)), and recall that the spherical cap discrepancy for the points in \mathcal{E}_n is defined by

$$(6.2) \quad \Delta_3(n) = \sup_{\substack{x \in S^2 \\ 0 < r \leq 2}} \left| \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_n : |\hat{\lambda} - x| \leq r \right\} - \frac{r^2}{4} \right|.$$

Lemma 6.1. *We have*

$$(6.3) \quad F_3(s) = \frac{s^2}{4} + O(\Delta_3(n)).$$

Proof. The estimate (6.3) follows immediately from the definition of spherical cap discrepancy, since

$$F_3(s) = \frac{1}{N} \sum_{\lambda' \in \mathcal{E}_n} \# \left\{ \lambda \in \mathcal{E}_n : 0 < \|\hat{\lambda} - \hat{\lambda}'\| \leq s \right\} = \frac{s^2}{4} + O(\Delta_3(n)).$$

\square

The discrepancy $\Delta_3(n)$ satisfies $\Delta_3(n) \leq n^{-\eta}$ for some small $\eta > 0$, see [8]. We are now in a position to prove Theorem 1.3:

Proof of Theorem 1.3. By (4.7) we have

$$(6.4) \quad \mathcal{V}(X_{f_n, r}) = 16\pi^3 r^6 \frac{1}{N^2} \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} h_3 \left(T \|\hat{\lambda} - \hat{\lambda}'\| \right) + O \left(r^6 N^{-1/4+\epsilon} \right).$$

For the summation in (6.4) we have,

$$\frac{1}{N^2} \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} h_3 \left(T \|\hat{\lambda} - \hat{\lambda}'\| \right) = \int_0^2 h_3(Ts) dF_3(s).$$

Thus, integrating by parts and using Lemma 6.1,

$$(6.5) \quad \begin{aligned} \frac{1}{N^2} \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} h_3 \left(T \|\hat{\lambda} - \hat{\lambda}'\| \right) &= h_3(2T) F_3(2) - \int_0^2 F_3(s) dh_3(Ts) \\ &= h_3(2T) F_3(2) - \frac{1}{4} \int_0^2 s^2 dh_3(Ts) + Err(X_{f_n, r}) \end{aligned}$$

where

$$(6.6) \quad Err(X_{f_n, r}) \ll \Delta_3(n) \int_0^2 T |h'_3(Ts)| ds.$$

Note that $h_3(s) \ll s^{-4}$ as $s \rightarrow \infty$. Thus, integrating by parts, the main term on the r.h.s of (6.5) satisfies

$$(6.7) \quad h_3(2T) F_3(2) - \frac{1}{4} \int_0^2 s^2 dh_3(Ts) = \frac{1}{2} \int_0^2 s \cdot h_3(Ts) ds + O(T^{-4}),$$

so that

$$(6.8) \quad \int_0^2 s \cdot h_3(Ts) \, ds = \frac{1}{T^2} \int_0^{2T} s \cdot h_3(s) \, ds = \frac{1}{T^2} \int_0^\infty s \cdot h_3(s) \, ds + O(T^{-4}).$$

A direct computation shows that

$$(6.9) \quad \int_0^\infty s \cdot h_3(s) \, ds = \frac{1}{2\pi^3} \int_0^\infty \frac{1}{s^3} \left(\frac{\sin s}{s} - \cos s \right)^2 \, ds = (2\pi)^{-3},$$

and therefore, substituting (6.9) into (6.8) and then into (6.7) we get

$$(6.10) \quad h_3(2T) F_3(2) - \frac{1}{4} \int_0^2 s^2 \, dh_3(Ts) = \frac{1}{16\pi^3} T^{-2} + O(T^{-4}).$$

Note that $h'_3(s) \ll (1 + s^4)^{-1}$. Thus,

$$(6.11) \quad \int_0^2 T |h'_3(Ts)| \, ds = \int_0^{2T} |h'_3(s)| \, ds \leq \int_0^\infty |h'_3(s)| \, ds < \infty$$

and therefore, substituting (6.11) into (6.6) we obtain

$$(6.12) \quad Err(X_{f_n,r}) = O(\Delta_3(n)).$$

Substituting (6.12) into (6.10) and finally into (6.5) we obtain (1.21). \square

Note that by (6.12),

$$\sup_{\substack{r > r_0 \\ f_n \in \mathcal{B}_n}} \left| \frac{\mathcal{V}(X_{f_n,r})}{r^6} - T^{-2} \right| = O(T_0^{-4} + n^{-\eta})$$

for every $n \neq 0, 4, 7$ (8), and in particular

$$\mathcal{V}(X_{f_n,r}) = o(r^6)$$

uniformly for $r > r_0$ for every $n \neq 0, 4, 7$ (8).

7. PROOFS OF THEOREM 1.5 AND THEOREM 1.6

Proof of theorems 1.5 and 1.6, upper bounds. By substituting the bound $|c_\lambda|^2 \leq N^{-1+\epsilon}$ into (4.5), we have

$$(7.1) \quad \mathcal{V}(X_{f_n,r}) \ll r^4 N^{-1+\epsilon} \sum_{\lambda_0 \in \mathcal{E}_n} |c_{\lambda_0}|^2 \sum_{\lambda \in \mathcal{E}_n} h_2(r \|\lambda - \lambda_0\|).$$

By Lemma 4.3 and by (5.7), we have

$$(7.2) \quad \begin{aligned} \frac{1}{N} \sum_{\lambda \in \mathcal{E}_n} h_2(r \|\lambda - \lambda_0\|) &= \int_0^2 h_2(Ts) \, dF_{\lambda_0}(s) \\ &= h_2(2T) - \frac{1}{4N} + O\left(\int_0^2 (s + \Delta(n)) T |h'_2(Ts)| \, ds\right) \\ &= O\left(T^{-1} + (\log n)^{-\frac{1}{2} \log \frac{\pi}{2} + \epsilon}\right) \end{aligned}$$

for n satisfying the hypothesis $\mathcal{D}(n, \epsilon)$. Substituting (7.2) in (7.1), we get the upper bound in Theorem 1.5. The upper bound (1.26) in Theorem 1.6 follows along similar lines. \square

We now turn to proving the claimed lower bounds for the variance of $X_{f_n,r}$. First, we need the following lemma, proved at the end of section 7:

Lemma 7.1. (1) Let $\{x_m\}_{m=1}^M$ be M points on the unit circle S^1 . For every $1 < T < M/2$ we have

$$\#\{x_i \neq x_j : |x_i - x_j| \leq 1/T\} \gg M^2/T.$$

(2) Let $\{x_m\}_{m=1}^M$ be M points on the unit sphere S^2 . For every $1 < T < \sqrt{M}/2$ we have

$$\#\{x_i \neq x_j : |x_i - x_j| \leq 1/T\} \gg M^2/T^2.$$

We are now in a position to prove the lower bounds (1.25), (1.26) of Theorem 1.5 and Theorem 1.6:

Proof of Theorem 1.5 and Theorem 1.6, lower bounds assuming Lemma 7.1. For $d = 2$, we let

$$R = \#\left\{\lambda \in \mathcal{E}_n : |c_\lambda|^2 \geq \frac{1}{2N}\right\},$$

so that

$$1 = \sum_{\lambda \in \mathcal{E}_n} |c_\lambda|^2 = \sum_{\lambda \in R} |c_\lambda|^2 + \sum_{\lambda \notin R} |c_\lambda|^2 \leq N^{-1+\epsilon} \cdot \#R + 1/2,$$

and hence $\#R \geq 2N^{1-\epsilon}$. By the second part of Lemma 4.3, for $c > 0$ sufficiently small we have

$$\begin{aligned} \mathcal{V}(X_{f_n, r}) &= 8\pi^2 r^4 \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} |c_\lambda|^2 |c_{\lambda'}|^2 h_2\left(T \|\hat{\lambda} - \hat{\lambda}'\|\right) \gg r^4 N^{-2} \sum_{\lambda \neq \lambda' \in R} h_2\left(T \|\hat{\lambda} - \hat{\lambda}'\|\right) \\ &\gg r^4 N^{-2} \cdot \#\left\{\lambda \neq \lambda' \in R : \|\hat{\lambda} - \hat{\lambda}'\| \leq c/T\right\}. \end{aligned}$$

By the first part of Lemma 7.1,

$$\mathcal{V}(X_{f_n, r}) \gg r^4 N^{-2} (\#R)^2 T^{-1} \gg r^4 N^{-2\epsilon} T^{-1}.$$

The lower bound (1.26) of Theorem 1.6 follows along the same lines as the above, this time using the second part of Lemma 7.1 in place of the first one. \square

Note that in the proof of the lower bound in Theorem 1.5 we have used the abundance of close-by pairs of lattice points with $|c_\lambda|^2 \geq \frac{1}{2N}$; in the absence of such close-by lattice points, the bound does not hold. For example, for $d = 2$, fix $\lambda_0 \in \mathcal{E}_n$ and let $|c_{\pm\lambda_0}|^2 = 1/2$ and $c_\lambda = 0$ for every $\lambda \neq \pm\lambda_0$. Then

$$\mathcal{V}(X_{f_n, r}) = 4\pi^2 r^4 h_2(2T) \ll r^4 T^{-3}.$$

Proof of Lemma 7.1. For the first part of Lemma 7.1, divide S^1 into $k = O(T)$ arcs I_1, I_2, \dots, I_k of length $< 1/T$. For every $1 \leq j \leq k$, let $n_j = \#\{m : x_m \in I_j\}$, so $\sum_{j=1}^k n_j = M$. By the Cauchy-Schwarz inequality,

$$M^2 = \left(\sum_{j=1}^k n_j\right)^2 \leq k \sum_{j=1}^k n_j^2 \ll T \sum_{j=1}^k n_j^2.$$

Thus,

$$\begin{aligned} \#\{x_i \neq x_j : |x_i - x_j| \leq 1/T\} &= \#\{x_i, x_j : |x_i - x_j| \leq 1/T\} - M \\ &\gg \sum_{j=1}^k n_j^2 - M \gg M^2/T - M \gg M^2/T. \end{aligned}$$

The second part of Lemma 7.1 is proved similarly. \square

8. RESTRICTED AVERAGES

8.1. Restricted moments. For $d = 2$, most of our principal results above are also valid in the more difficult scenario where x is drawn in $B_{x_0}(\rho)$ for some $x_0 \in \mathbb{T}^2$ and $\rho \gg n^{-1/2+o(1)}$. In this case, the restricted moments are: expectation

$$(8.1) \quad \mathbb{E}_{x_0, \rho}[X_{f_n, r}] = \frac{1}{\text{Vol}(B_{x_0}(\rho))} \int_{B_{x_0}(\rho)} X_{f_n, r; x} dx,$$

higher centred moments

$$(8.2) \quad \mathbb{E}_{x_0, \rho}[(X_{f_n, r} - \mathbb{E}_{x_0, \rho}[X_{f_n, r}])^k] = \frac{1}{\text{Vol}(B_{x_0}(\rho))} \int_{B_{x_0}(\rho)} (X_{f_n, r; x} - \mathbb{E}_{x_0, \rho}[X_{f_n, r}])^k dx, \quad k \geq 2,$$

and in particular the variance

$$(8.3) \quad \mathcal{V}_{x_0, \rho}(X_{f_n, r}) = \mathbb{E}_{x_0, \rho}[(X_{f_n, r} - \mathbb{E}_{x_0, \rho}[X_{f_n, r}])^2].$$

We reinterpret the statement of Granville-Wigman's [15, Theorem 1.2] as evaluating the expected mass

$$\mathbb{E}_{x_0, \rho}[X_{f_n, r}] \sim \pi r^2,$$

valid for almost all $n \in S_2$, uniformly for $\rho \gg n^{-1/2+o(1)}$, $x_0 \in \mathbb{T}^2$, and $r > 0$ (see the first part of Lemma 8.4).

8.2. Quasi-correlations. For the restricted moments (8.2) of $X_{f_n, r}$ one also needs to cope with *quasi-correlations*, i.e. tuples $(\lambda^1, \dots, \lambda^l) \in \mathcal{E}_n^l$ with the sum $\sum_{i=1}^l \lambda^i$ unexpectedly small, e.g. given a (small) fixed number $\delta > 0$,

$$(8.4) \quad \left\| \sum_{i=1}^l \lambda^i \right\| < n^{1/2-\delta};$$

unlike the correlations (2.3), here there are no congruence obstructions, so that (8.4) makes sense with l odd or even.

Definition 8.1 (Quasi-correlations, cf. [2, Definition 1.3]).

- (1) For $n \in S_2$, $l \in \mathbb{Z}_{\geq 2}$, and $0 < K = K(n) < l \cdot n^{1/2}$ define the set of length- l spectral quasi-correlations

$$(8.5) \quad \mathcal{C}_n(l; K) = \left\{ (\lambda^1, \dots, \lambda^l) \in \mathcal{E}_n^l : 0 < \left\| \sum_{j=1}^l \lambda^j \right\| \leq K \right\}.$$

- (2) Given $\delta > 0$ we say that $n \in S_2$ satisfies the (l, δ) -separateness hypothesis $\mathcal{A}(n; l, \delta)$ if

$$(8.6) \quad \mathcal{C}_n(l; n^{1/2-\delta}) = \emptyset.$$

For example, $\mathcal{A}(n; 2, \delta)$ is equivalent to the aforementioned Bourgain-Rudnick separateness, satisfied [9, Lemma 5] by a density 1 sequence $S'_2 \subseteq S_2$. More generally, it was shown in the forthcoming paper [1], that for every $\delta > 0$ and $l \geq 2$, the assumption $\mathcal{A}(n; l, \delta)$ is satisfied by generic $n \in S'_2(l, \delta)$, and hence a standard diagonal argument yields a density 1 sequence $S'_2 \subseteq S_2$ so that $\mathcal{A}(n; l, \delta)$ is satisfied for all $l \geq 2$ and $\delta > 0$ for $n \in S'_2$ sufficiently large.

Theorem 8.2 ([1]). *For every fixed $l \geq 2$ and $\delta > 0$ there exist a set $S'_2(l, \delta) \subseteq S_2$ such that:*

- (1) *The set $S'_2(l, \delta)$ has density 1 in S_2 .*

(2) For every $n \in S'_2(l, \delta)$ the length- l spectral quasi-correlation set

$$\mathcal{C}_n(l; n^{1/2-\delta}) = \emptyset$$

is empty, i.e., the (l, δ) -separateness hypothesis $\mathcal{A}(n; l, \delta)$ is satisfied.

8.3. A version of Theorem 2.5 with restricted averages. We have the following analogue of Theorem 2.5:

Theorem 8.3. Let $\delta > 0$, and $0 < \epsilon < \delta/5$.

(1) If $S'_2 \subseteq S_2$ is a sequence satisfying the hypotheses $\mathcal{D}(n, \epsilon/2)$, $\mathcal{A}(n; 2, \epsilon)$, and $\mathcal{A}(n; 4, \epsilon)$ for all $n \in S'_2$, then in the setting of Theorem 2.5 part (1),

$$\mathcal{V}_{x_0, \rho}(X_{f_n, r}) \sim \frac{16}{3\pi \cos^2 \theta_{f_n}} r^4 T^{-1}$$

uniformly for all $x_0 \in \mathbb{T}^2$, $n^{-1/2+\delta} \leq \rho \leq 1$ and $r_0 < r < r_1$, and $f_n \in \mathcal{F}_1(n; T(n), \eta(n))$.

(2) Let $k \geq 3$ be an integer. If $S'_2 \subseteq S_2$ is a sequence satisfying the length- $2k$ diagonal domination assumption and the hypotheses $\mathcal{D}(n, \epsilon)$, $\mathcal{A}(n; 2, \epsilon)$, $\mathcal{A}(n; 4, \epsilon)$, and $\mathcal{A}(n; 2k, \epsilon)$ for all $n \in S'_2$, then in the setting of Theorem 2.5 part (2),

$$\mathbb{E}_{x_0, \rho} [\hat{X}_{f_n, r}^k] \rightarrow \mathbb{E}[Z^k]$$

uniformly for $x_0 \in \mathbb{T}^2$, $r_0 < r < r_1$, $n^{-1/2+\delta} \leq \rho \leq 1$, and $f_n \in \mathcal{F}_2(n; T(n), \eta(n))$, where $Z \sim N(0, 1)$ is the standard Gaussian variable.

Theorem 8.3 follows along similar lines as the proof of Theorem 2.5, where we use the expressions for the restricted moments below (cf. equation (4.1), Lemma 4.2 and equation (5.3)). We remark that Theorem 1.5 can also be extended to $\mathcal{V}_{x_0, \rho}(X_{f_n, r})$, however the lower bound will only hold for a generic $n \in S_2$.

Lemma 8.4 (Expectation and variance, $d = 2$, x drawn in shrinking discs). For $d = 2$ let $0 < \delta < 1/2$, $0 < \epsilon < \delta/5$, and $S'_2 \subseteq S_2$.

(1) If $n \in S'_2$ satisfy the hypothesis $\mathcal{A}(n; 2, \epsilon)$, then

$$\mathbb{E}_{x_0, \rho} [X_{f_n, r}] = \pi r^2 + O\left(r^2 n^{-\frac{3}{5}\delta + 3\epsilon}\right)$$

uniformly for $x_0 \in \mathbb{T}^2$, $n^{-1/2+\delta} \leq \rho \leq 1$ and $r > 0$.

(2) If $n \in S'_2$ satisfy the hypotheses $\mathcal{A}(n; 2, \epsilon)$ and $\mathcal{A}(n; 4, \epsilon)$, then

$$\mathcal{V}_{x_0, \rho}(X_{f_n, r}) = 8\pi^2 r^4 \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} |c_\lambda|^2 |c_{\lambda'}|^2 h_2(r \|\lambda - \lambda'\|) + O\left(r^4 n^{-\frac{3}{5}\delta + 4\epsilon}\right)$$

uniformly for $x_0 \in \mathbb{T}^2$, $n^{-1/2+\delta} \leq \rho \leq 1$ and $r > 0$.

Lemma 8.5 (Higher moments, $d = 2$, x drawn in shrinking discs). For $d = 2$ let $k \geq 3$, $0 < \delta < 1/2$, $0 < \epsilon < \delta/5$, and $S'_2 \subseteq S_2$ satisfying $\mathcal{A}(2; n, \epsilon)$, $\mathcal{A}(4; n, \epsilon)$, and $\mathcal{A}(n; 2k, \epsilon)$ for every $n \in S'_2$. We have

$$\begin{aligned} \mathbb{E}_{x_0, \rho} [\hat{X}_{f_n, r}^k] &= (2\pi)^k r^{2k} \mathcal{V}_{x_0, \rho}(X_{f_n, r})^{-k/2} \sum_{\substack{\forall 1 \leq i \leq k, \lambda_i \neq \lambda'_i \in \mathcal{E}_n \\ \sum_{i=1}^k (\lambda_i - \lambda'_i) = 0}} \prod_{j=1}^k c_{\lambda_j} \overline{c_{\lambda'_j}} g_2(r \|\lambda_j - \lambda'_j\|) \\ &\quad + O\left(\mathcal{V}_{x_0, \rho}(X_{f_n, r})^{-k/2} r^{2k} n^{-\frac{3}{5}\delta + 4\epsilon}\right) \end{aligned}$$

uniformly for $x_0 \in \mathbb{T}^2$, $n^{-1/2+\delta} \leq \rho \leq 1$ and $r > 0$.

8.4. Proofs of Lemma 8.4 and Lemma 8.5.

Proof of Lemma 8.4, 1st part. We have

$$(8.7) \quad \mathbb{E}_{x_0, \rho} [X_{f_n, r}] = \frac{1}{\pi \rho^2} \int_{B_{x_0}(\rho)} \int_{B_x(r)} f_n(y)^2 \, dy \, dx.$$

Granville-Wigman's [15, Theorem 1.2] asserts that for $\epsilon_1 > \epsilon_2 > 0$, $0 < \epsilon_3 < \epsilon_1 - \epsilon_2$ and $n \in S_2$ satisfying $\mathcal{A}(n; 2, \epsilon_2)$, we have

$$(8.8) \quad \int_{B_x(r)} f_n(y)^2 \, dy = \pi r^2 \left(1 + O \left(n^{-3\epsilon_3/2} \right) \right)$$

uniformly in $x \in \mathbb{T}^2$ and $r > n^{-1/2+\epsilon_1}$. If $r > n^{-1/2+\frac{2}{5}\delta}$, then by substituting (8.8) with $\epsilon_1 = \frac{2}{5}\delta$, $\epsilon_2 = \epsilon$ and $\epsilon_3 = \frac{2}{5}\delta - 2\epsilon$ into (8.7), we have

$$\mathbb{E}_{x_0, \rho} [X_{f_n, r}] = \pi r^2 \left(1 + O \left(n^{-\frac{3}{2}(\frac{2}{5}\delta - 2\epsilon)} \right) \right)$$

for every ρ .

Otherwise, note that

$$\mathbb{E}_{x_0, \rho} [X_{f_n, r}] = \frac{1}{\pi \rho^2} \int_{B_{x_0}(\rho+r)} f_n(y)^2 \int_{B_{x_0}(\rho) \cap B_y(r)} dx \, dy,$$

so

$$(8.9) \quad \frac{r^2}{\rho^2} \int_{B_{x_0}(\rho-r)} f_n(y)^2 \, dy \leq \mathbb{E}_{x_0, \rho} [X_{f_n, r}] \leq \frac{r^2}{\rho^2} \int_{B_{x_0}(\rho+r)} f_n(y)^2 \, dy.$$

Since $r/\rho \leq n^{-\frac{3}{5}\delta}$, we can use (8.8) with $\epsilon_1 = \delta$, $\epsilon_2 = \epsilon$ and $\epsilon_3 = \delta - 2\epsilon$ to deduce that

$$(8.10) \quad \int_{B_{x_0}(\rho \pm r)} f_n(y)^2 \, dy = \pi \rho^2 \left(1 + O \left(n^{-\frac{3}{5}\delta} \right) \right),$$

and the statement of the first part of Lemma 8.4 follows upon substituting (8.10) into (8.9). \square

Proof of Lemma 8.4, 2nd part. We have

$$\mathcal{V}_{x_0, \rho}(X_{f_n, r}) = \frac{1}{\pi \rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 \, dy - \mathbb{E}_{x_0, \rho} [X_{f_n, r}] \right)^2 dx.$$

By (4.3),

$$\begin{aligned}
& \frac{1}{\pi\rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 dy - \pi r^2 \right)^2 dx \\
&= 4\pi^2 r^4 \sum_{\substack{\lambda, \lambda', \lambda'', \lambda''' \in \mathcal{E}_n \\ \lambda \neq \lambda' \\ \lambda'' \neq \lambda'''}} c_\lambda \overline{c_{\lambda'}} c_{\lambda''} \overline{c_{\lambda'''}} g_2(r \|\lambda - \lambda'\|) g_2(r \|\lambda'' - \lambda'''\|) \\
&\times \frac{1}{\pi\rho^2} \int_{B_{x_0}(\rho)} e(\langle x, \lambda - \lambda' + \lambda'' - \lambda''' \rangle) dx \\
&= 8\pi^2 r^4 \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} |c_\lambda|^2 |c_{\lambda'}|^2 g_2(r \|\lambda - \lambda'\|)^2 \\
&+ 8\pi^2 r^4 \sum_{\substack{\lambda, \lambda', \lambda'', \lambda''' \in \mathcal{E}_n \\ \lambda \neq \lambda' \\ \lambda'' \neq \lambda''' \\ \lambda - \lambda' + \lambda'' - \lambda''' \neq 0}} c_\lambda \overline{c_{\lambda'}} c_{\lambda''} \overline{c_{\lambda'''}} g_2(r \|\lambda - \lambda'\|) g_2(r \|\lambda'' - \lambda'''\|) \\
&\times e(\langle x_0, \lambda - \lambda' + \lambda'' - \lambda''' \rangle) g_2(\rho \|\lambda - \lambda' + \lambda'' - \lambda'''\|).
\end{aligned}$$

By the hypothesis $\mathcal{A}(n; 4, \epsilon)$ and Lemma 4.3, we have

$$\begin{aligned}
& \sum_{\substack{\lambda, \lambda', \lambda'', \lambda''' \in \mathcal{E}_n \\ \lambda \neq \lambda' \\ \lambda'' \neq \lambda''' \\ \lambda - \lambda' + \lambda'' - \lambda''' \neq 0}} c_\lambda \overline{c_{\lambda'}} c_{\lambda''} \overline{c_{\lambda'''}} g_2(r \|\lambda - \lambda'\|) g_2(r \|\lambda'' - \lambda'''\|) \\
&\times e(\langle x_0, \lambda - \lambda' + \lambda'' - \lambda''' \rangle) g_2(\rho \|\lambda - \lambda' + \lambda'' - \lambda'''\|) \\
&\ll \left(\sum_{\lambda \in \mathcal{E}_n} |c_\lambda| \right)^4 \frac{1}{(n^{\delta-\epsilon})^{3/2}} \ll N^2 n^{-\frac{3}{2}(\delta-\epsilon)} \ll n^{-\frac{3}{2}\delta+2\epsilon}.
\end{aligned}$$

Next, note that

$$(8.11) \quad \int_{B_x(r)} f_n(y)^2 dy - \pi r^2 = 2\pi r^2 \sum_{\lambda \neq \lambda' \in \mathcal{E}_n} c_\lambda \overline{c_{\lambda'}} g_2(r \|\lambda - \lambda'\|) \ll r^2 \left(\sum_{\lambda \in \mathcal{E}_n} |c_\lambda| \right)^2 \ll N r^2.$$

By (8.11) and the first part of Lemma 8.4,

$$\mathcal{V}_{x_0, \rho}(X_{f_n, r}) = \frac{1}{\pi\rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 dy - \pi r^2 \right)^2 dx + O\left(r^4 n^{-\frac{3}{5}\delta+4\epsilon}\right)$$

and the statement of Lemma 4.2 follows. \square

Proof of Lemma 8.5. We have

$$\mathbb{E}_{x_0, \rho}[\hat{X}_{f_n, r}^k] = \mathcal{V}_{x_0, \rho}(X_{f_n, r})^{-k/2} \cdot \frac{1}{\pi\rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 dy - \mathbb{E}_{x_0, \rho}[X_{f_n, r}] \right)^k dx.$$

By (4.3), we have

$$\begin{aligned} \frac{1}{\pi \rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 \, dy - \pi r^2 \right)^k \, dx &= (2\pi)^k r^{2k} \sum_{\substack{\forall 1 \leq i \leq k, \lambda_i \neq \lambda'_i \in \mathcal{E}_n \\ \sum_{i=1}^k (\lambda_i - \lambda'_i) = 0}} \prod_{j=1}^k c_{\lambda_j} \overline{c_{\lambda'_j}} g_2(r \|\lambda_j - \lambda'_j\|) \\ &+ (2\pi)^k r^{2k} \sum_{\substack{\forall 1 \leq i \leq k, \lambda_i \neq \lambda'_i \in \mathcal{E}_n \\ \sum_{i=1}^k (\lambda_i - \lambda'_i) \neq 0}} \prod_{j=1}^k c_{\lambda_j} \overline{c_{\lambda'_j}} g_2(r \|\lambda_j - \lambda'_j\|) \\ &\times 2e \left(\left\langle x_0, \sum_{j=1}^k (\lambda_j - \lambda'_j) \right\rangle \right) g_2 \left(\rho \left\| \sum_{j=1}^k (\lambda_j - \lambda'_j) \right\| \right). \end{aligned}$$

By the hypothesis $\mathcal{A}(n; 2k, \epsilon)$,

$$\begin{aligned} \sum_{\substack{\forall 1 \leq i \leq k, \lambda_i \neq \lambda'_i \in \mathcal{E}_n \\ \sum_{i=1}^k (\lambda_i - \lambda'_i) \neq 0}} \prod_{j=1}^k c_{\lambda_j} \overline{c_{\lambda'_j}} g_2(r \|\lambda_j - \lambda'_j\|) e \left(\left\langle x_0, \sum_{j=1}^k (\lambda_j - \lambda'_j) \right\rangle \right) g_2 \left(\rho \left\| \sum_{j=1}^k (\lambda_j - \lambda'_j) \right\| \right) \\ \ll \left(\sum_{\lambda \in \mathcal{E}_n} |c_\lambda| \right)^{2k} \frac{1}{(n^{\delta-\epsilon})^{3/2}} \ll N^k n^{-\frac{3}{2}(\delta-\epsilon)} \ll n^{-\frac{3}{2}+2\epsilon}. \end{aligned}$$

By (8.11) and the first part of Lemma 8.4,

$$\begin{aligned} \mathbb{E}_{x_0, \rho} [\hat{X}_{f_n, r}^k] &= \mathcal{V}_{x_0, \rho} (X_{f_n, r})^{-k/2} \cdot \frac{1}{\pi \rho^2} \int_{B_{x_0}(\rho)} \left(\int_{B_x(r)} f_n(y)^2 \, dy - \pi r^2 \right)^k \, dx \\ &+ O \left(\mathcal{V}_{x_0, \rho} (X_{f_n, r})^{-k/2} r^{2k} n^{-\frac{3}{5}\delta + 4\epsilon} \right), \end{aligned}$$

and the statement of Lemma 8.5 follows. \square

9. PROOFS OF AUXILIARY LEMMAS

In this section we provide the proofs for lemmas 2.3, 4.1, and 4.7:

Proof of Lemma 2.3.

- (1) The upper bound is straightforward, and the lower bound follows from (1.8) by invoking the Cauchy-Schwarz inequality on (1.8).
- (2) By partial summation, for every $\lambda_0 \in \mathcal{E}_n$ we have

$$1 = \sum_{\lambda \in \mathcal{E}_n} |c_\lambda|^2 = N |c_{\lambda_0}|^2 + E,$$

where $|E| \leq V(\underline{v})$. Since λ_0 is arbitrary, we deduce that

$$[v]_\infty \leq 1 + V(\underline{v}).$$

- (3) Follows directly from parts 1 and 2 of this lemma. \square

Proof of Lemma 4.1. We have

$$(9.1) \quad \int_{B_x(r)} f_n(y)^2 dy = \int_{B_x(r)} \sum_{\lambda, \lambda' \in \mathcal{E}_n} c_\lambda \overline{c_{\lambda'}} e(\langle y, \lambda - \lambda' \rangle) dy$$

$$= \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d + \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_n \\ \lambda \neq \lambda'}} c_\lambda \overline{c_{\lambda'}} \int_{B_x(r)} e(\langle y, \lambda - \lambda' \rangle) dy.$$

Transforming the variables $y = rz + x$, we obtain

$$(9.2) \quad \int_{B_x(r)} e(\langle y, \lambda - \lambda' \rangle) dy = r^d e(\langle x, \lambda - \lambda' \rangle) \int_{B_0(1)} e(\langle z, r(\lambda - \lambda') \rangle) dz.$$

Note that

$$(9.3) \quad \int_{B_0(1)} e(\langle z, r(\lambda - \lambda') \rangle) dz = \frac{(2\pi)^{d/2} J_{d/2}(2\pi r \|\lambda - \lambda'\|)}{(2\pi r \|\lambda - \lambda'\|)^{d/2}},$$

and (4.3) follows upon substituting (9.3) into (9.2) and finally into (9.1). \square

Proof of Lemma 4.7. Let θ_λ be the angle between λ and λ' . Then

$$\begin{aligned} \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_n : \lambda \succeq \lambda', \|\hat{\lambda} - \hat{\lambda}'\| \leq s \right\} &= \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_n : \theta_\lambda \geq 0, \sqrt{2(1 - \cos \theta_\lambda)} \leq s \right\} \\ &= \frac{1}{N} \cdot \# \left\{ \lambda \in \mathcal{E}_n : \theta_\lambda \in [0, \arccos(1 - s^2/2)] \right\} \\ &= \frac{1}{2\pi} \arccos(1 - s^2/2) + O(\Delta(n)) \\ &= \frac{s}{2\pi} + O(s^3 + \Delta(n)) \end{aligned}$$

which is the statement (4.18) of Lemma 4.7. \square

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